Graph Theory : An Introduction

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Module V – Course Content

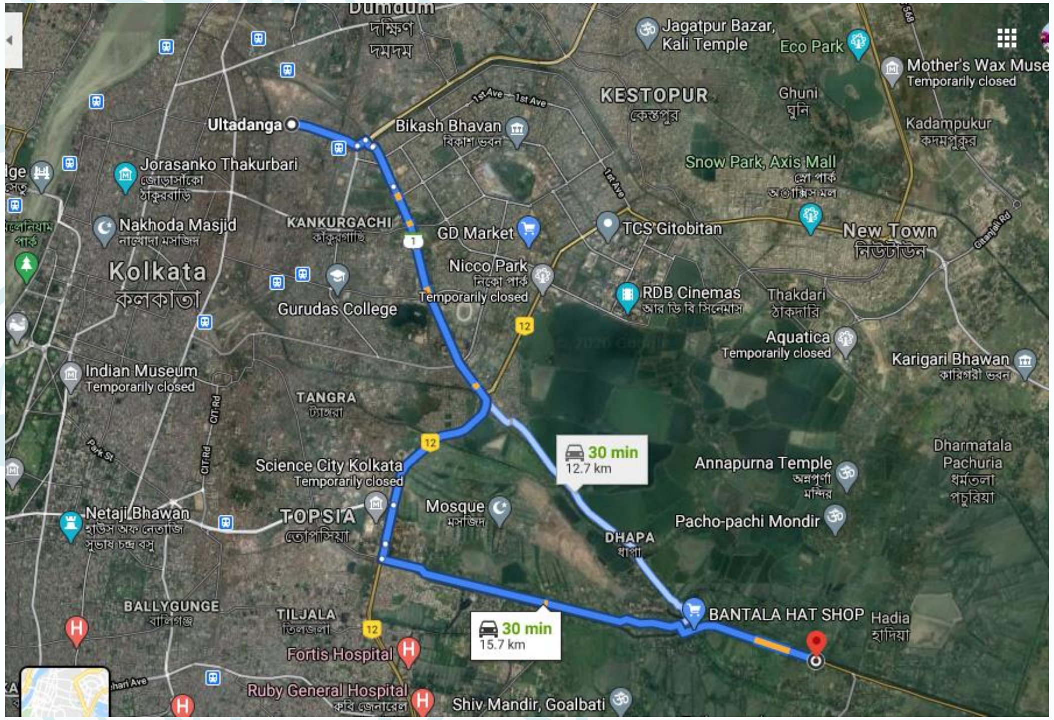
- Basic concept of graph
- Walk, Path, Circuit
- Euler and Hamiltonian graph
- Digraph
- Matrix representation: Incidence and Adjacency matrix
- Tree: Basic concept of tree
- Binary tree
- Spanning tree
- Kruskal and Prim's algorithm for finding the MST
- Dijkstra's Algorithm for finding the Shortest Path between nodes

References -

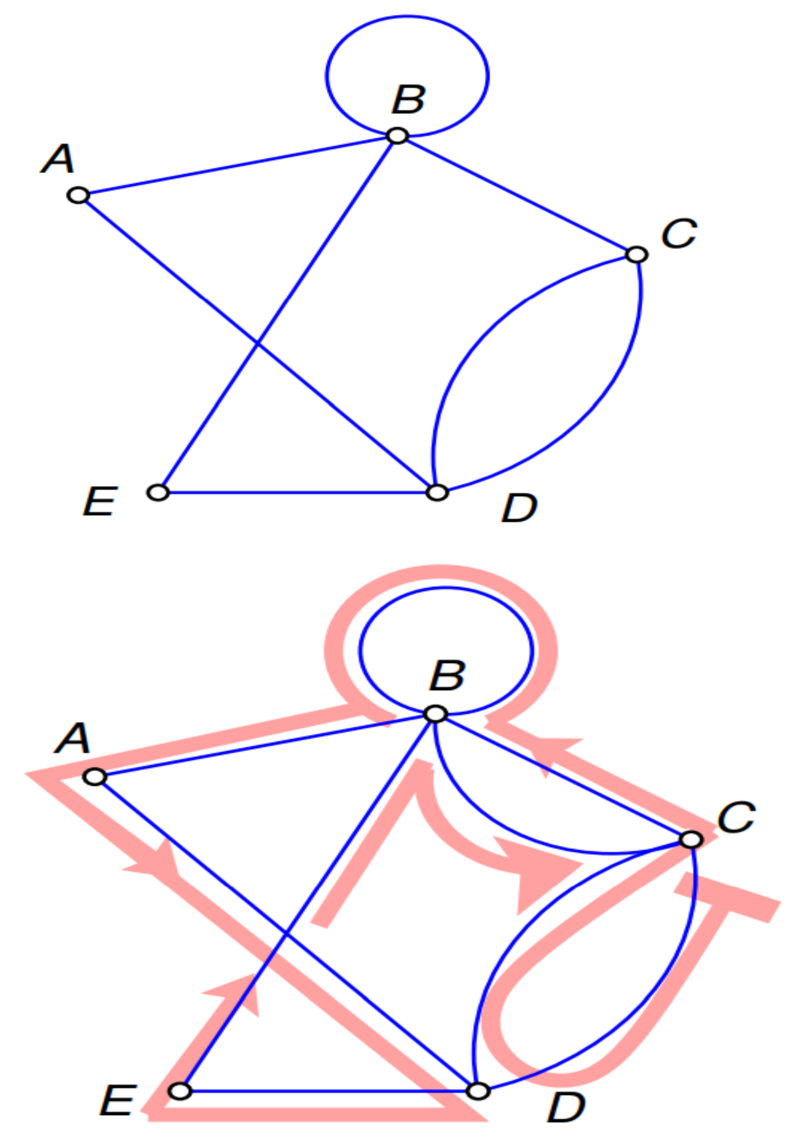
- Introduction to Graph Theory, by D.B. West
- Graph Theory with Applications to Engineering and Computer Science, by Narsingh Deo
- Graph Theory, by Reinhard Diestel
- Elements of Discrete Mathematics: A Complete Oriented Approach, by Liu & Mohapatra
- Discrete Mathematics and Its Applications with Combinatorics and Graph Theory, by Rosen
- https://www.geeksforgeeks.org/introduction-to-graphs/
- https://www.tutorialspoint.com/graph_theory/index.htm

Why??

Nothing But A Connected Nodes



- ❑ Everything in our world is inked :
- All cities are linked by maps (Google Map)
- Internet: One of the largest graph in the world. Pages are linked by hyperlinks on the internet
- Flight and Railway network
- Components of electric circuit etc..



An Euler circuit: CDCBBADEBC

Chinese Postman Problem or Route Inspection Problem

- A postman who has to deliver mails to houses along each of the streets in a particular housing estate and wants to minimize the distance he has to walk. Also he must return to his starting point. What is the ***shortest possible path in which he visits every street exactly once and returns to his starting point*** ?

✓ Eulerian Graph

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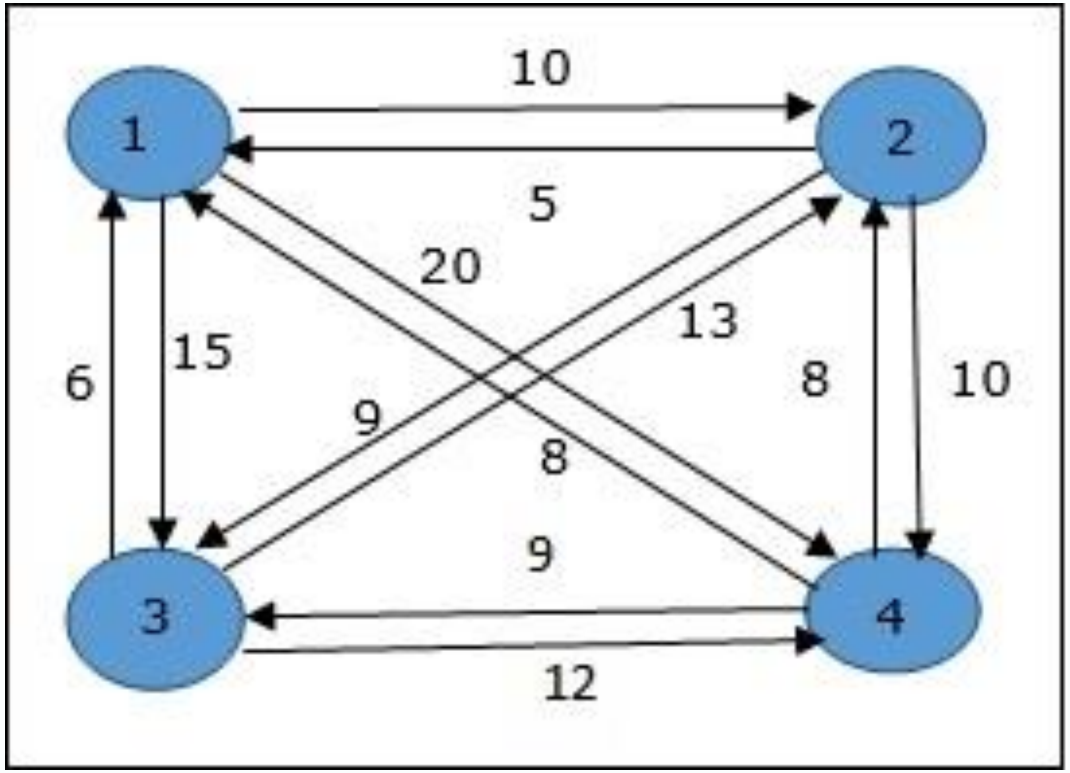
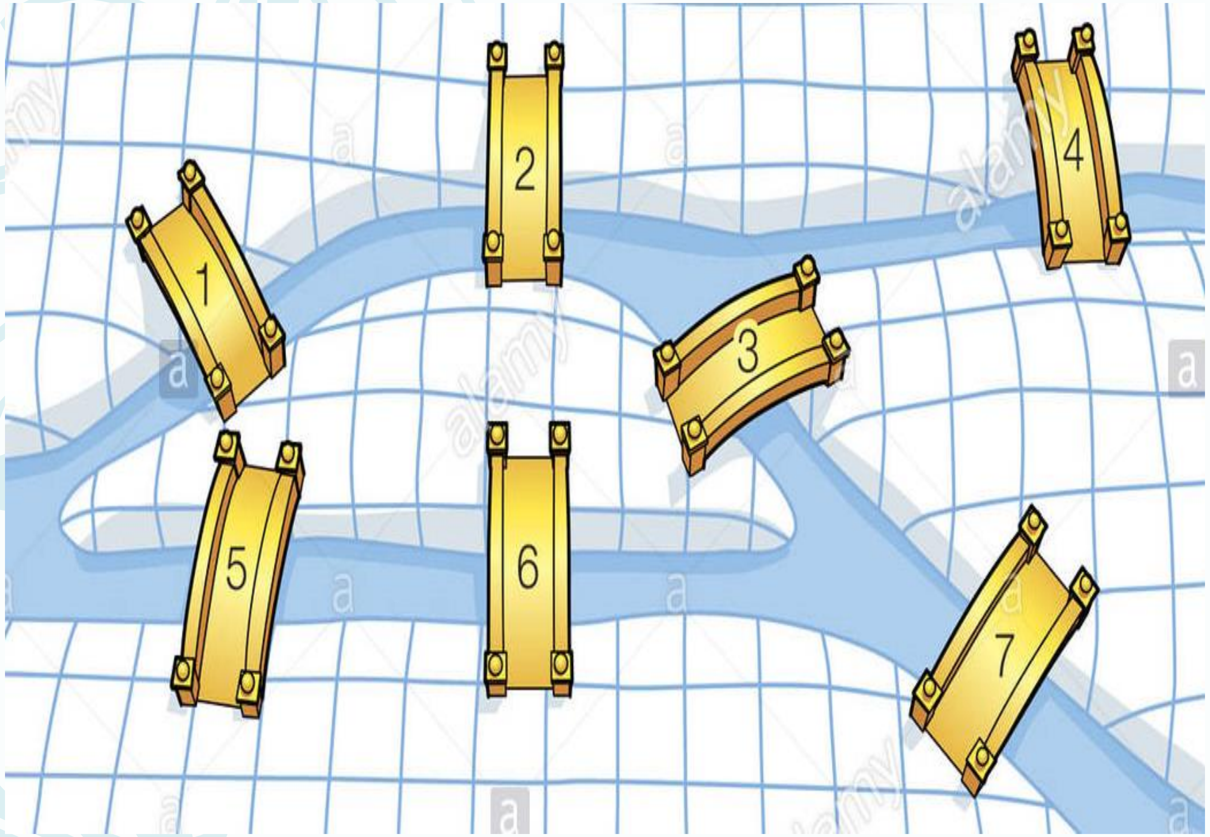
Traveling Salesman Problem



- A traveler needs to visit all the cities from a list, where ***distances between all the cities are known and each city should be visited just once.*** What is the shortest possible route that he visits each city exactly once and returns to the origin city?

✓ **Hamiltonian Graph**

Where, How, When ??
Konigsberg Bridge Problem



- Konigsberg is a city on the Pregal river in Prussia
- The city occupied two islands plus areas on both banks

Problem :

There are seven bridges to connect the two islands and the downstream parts of the town. Can one person start walking from his home, cross all seven bridges only once and visit every part of Konigsberg and return to home?

Solution – ***Impossible!!***

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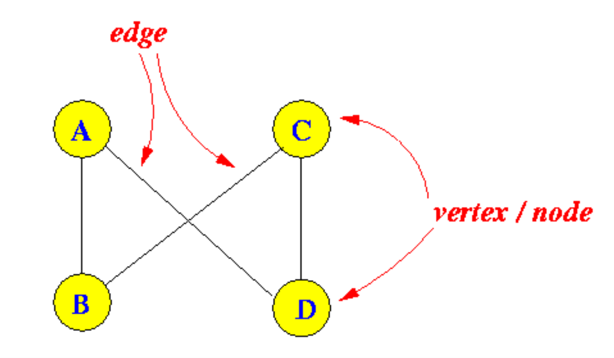
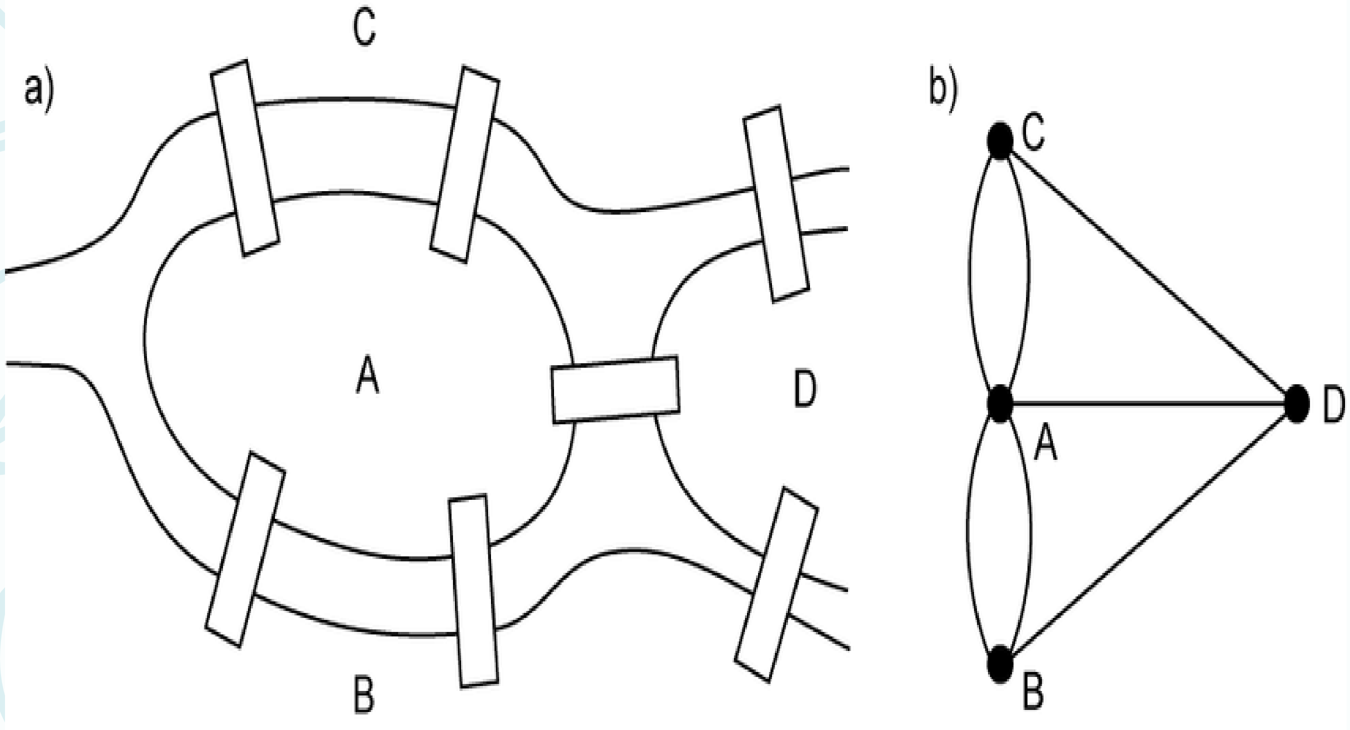
What??



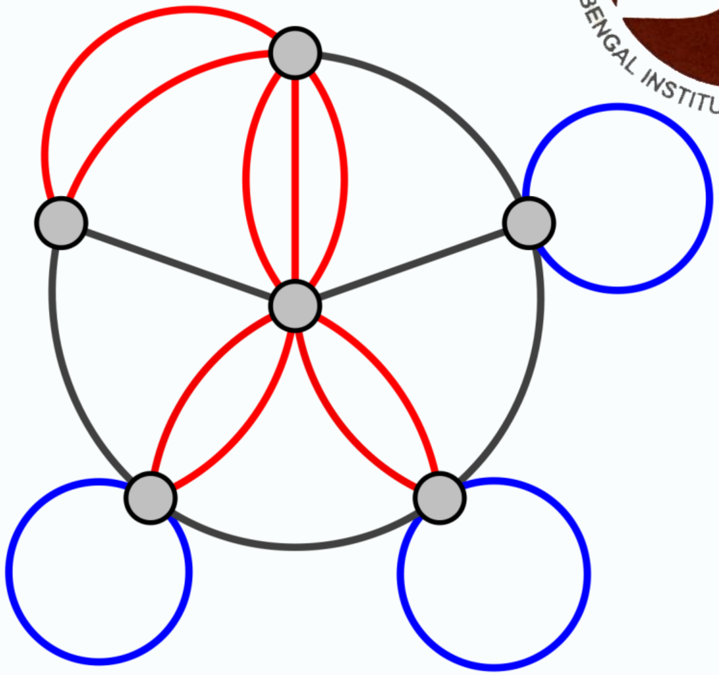


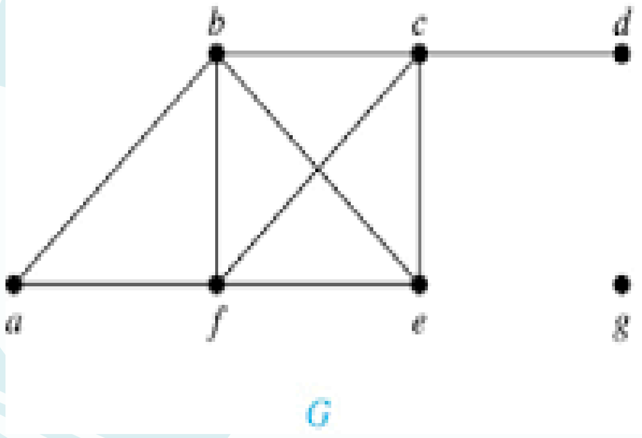
Definition -

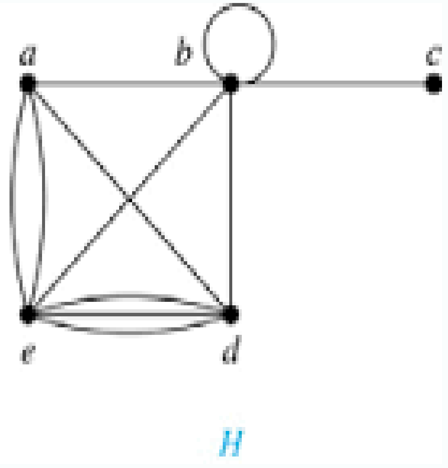
- A graph is an ordered pair of two sets, Vertex set and Edge set.
- G = (V, E), where V is the set of vertices and E is the set of edges.
- $V = \{v_1, v_2, \ldots\}$, $E = \{e_1, e_2, \ldots\}$
- Set of edges is defined as an into mapping from V X V to E, i.e. $f : V \times V \rightarrow E$ such that f(u, v) = e

Few Important Terminologies -

- Self Loop – A loop is an edge whose endpoints are equal
- Multiple or Parallel Edge – edges having the same pair of endpoints
- Simple Graph – a graph having no loops or multiple edges.
- Multi Graph – a graph having self loops or multiple edges.



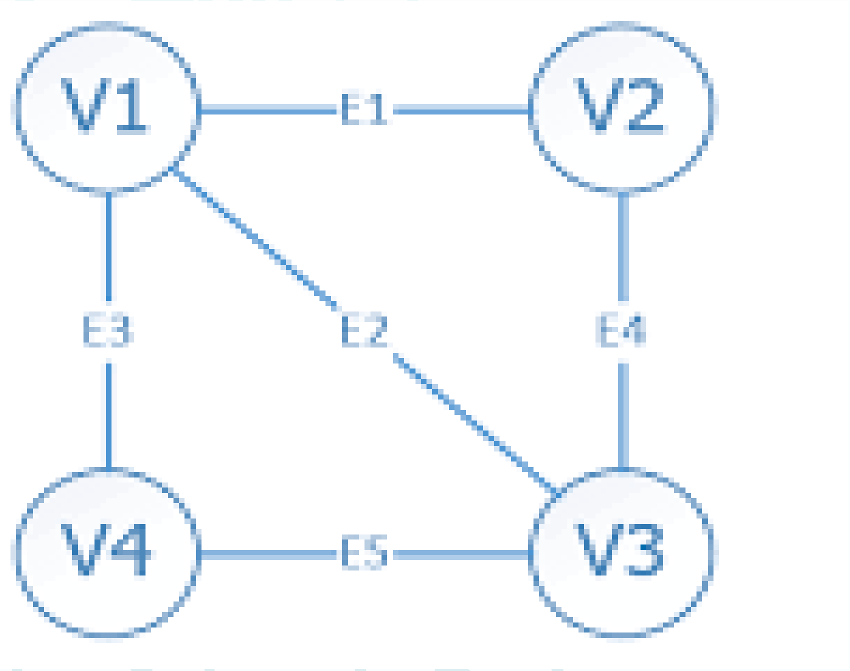




- Adjacent vertex – when u and v are endpoints of an edge, they are adjacent and are neighbors. In the graph G, vertex (a, b), (a, f),(b, c) etc. are adjacent vertices. But (a, c) is non-adjacent with each other.

Few Important Terminologies -

- Null graph – a graph whose edge set is empty.
- Isolated Vertex – a vertex having no incident edges.



- Incident Edges - Two edges are incident if they share a common vertex.

✓ Not only edges, but vertices can also be incident with an edge.

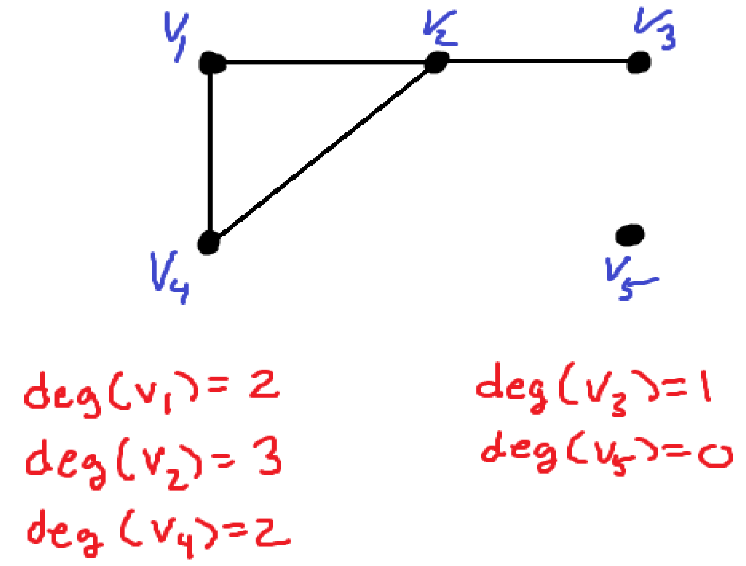
✓ A vertex is incident with an edge if the vertex is one of the endpoints of that edge.

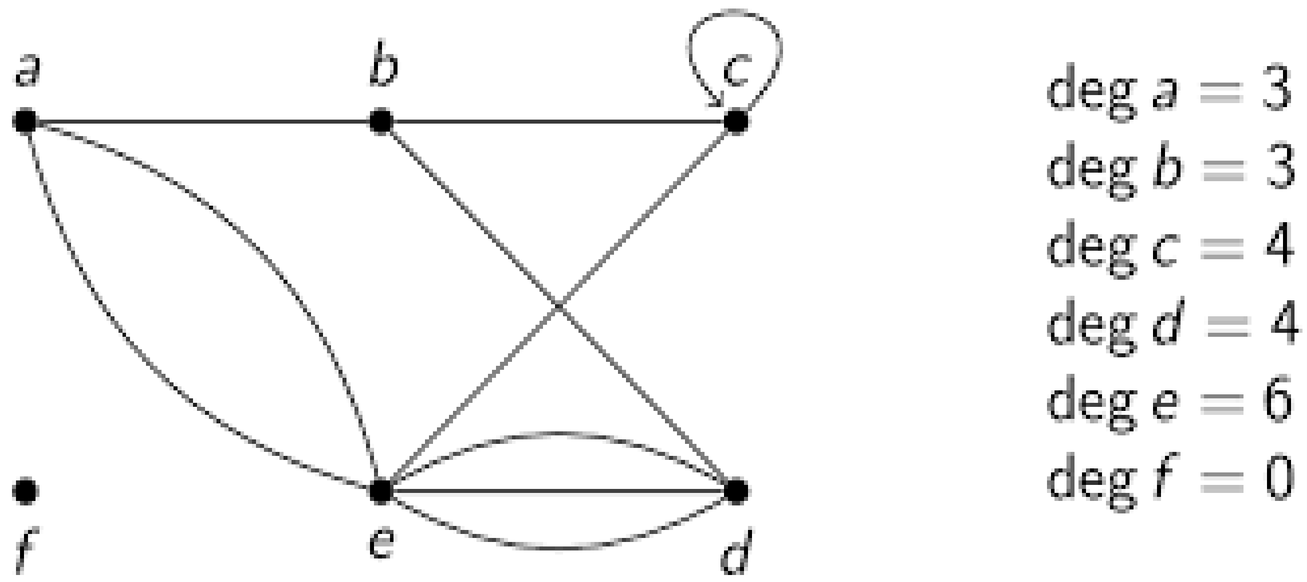
✓ Incident Edges – (E_1, E_2, E_3), (E_2,E_4,E_5), (E_1, E_4), (E_3,E_5)

✓ Adjacent Vertices – (V_1, V_2), (V_1,V_3), (V_1, V_4), (V_2, V_3), (V_3, V_4)

Degree of a Vertex -

- Definition – The degree of a vertex v in a graph G is the number of edges incident to v, except that each loop at v counts twice.





Notation –

a. Degree of a vertex v - $d_G(v)$ or d(v) or deg(v)
b. Maximum degree - $\Delta(G)$, Minimum degree - $\delta(G)$
c. Neighborhood of v - $N_G(v)$ or N(v) which is the set of vertices adjacent to v

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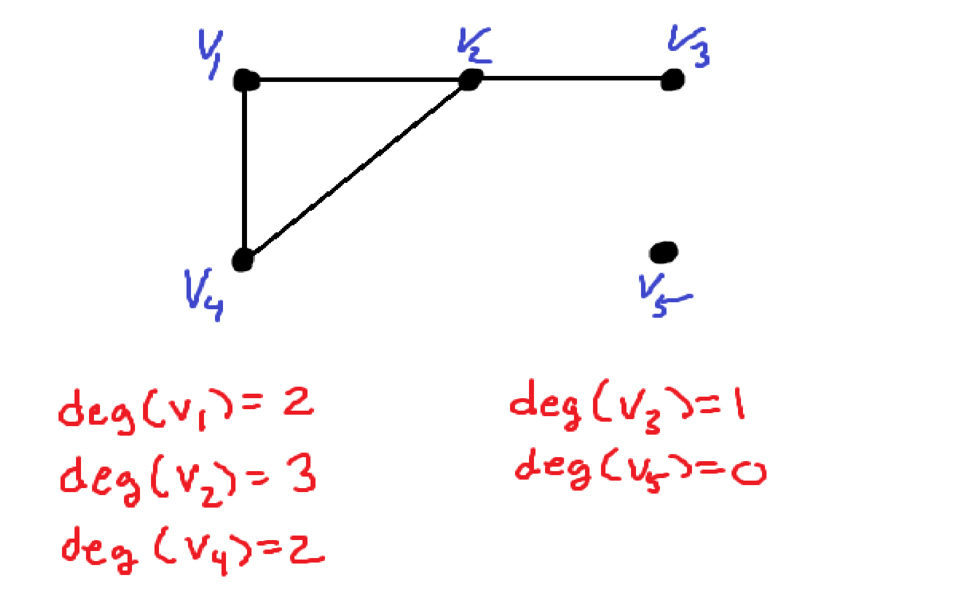
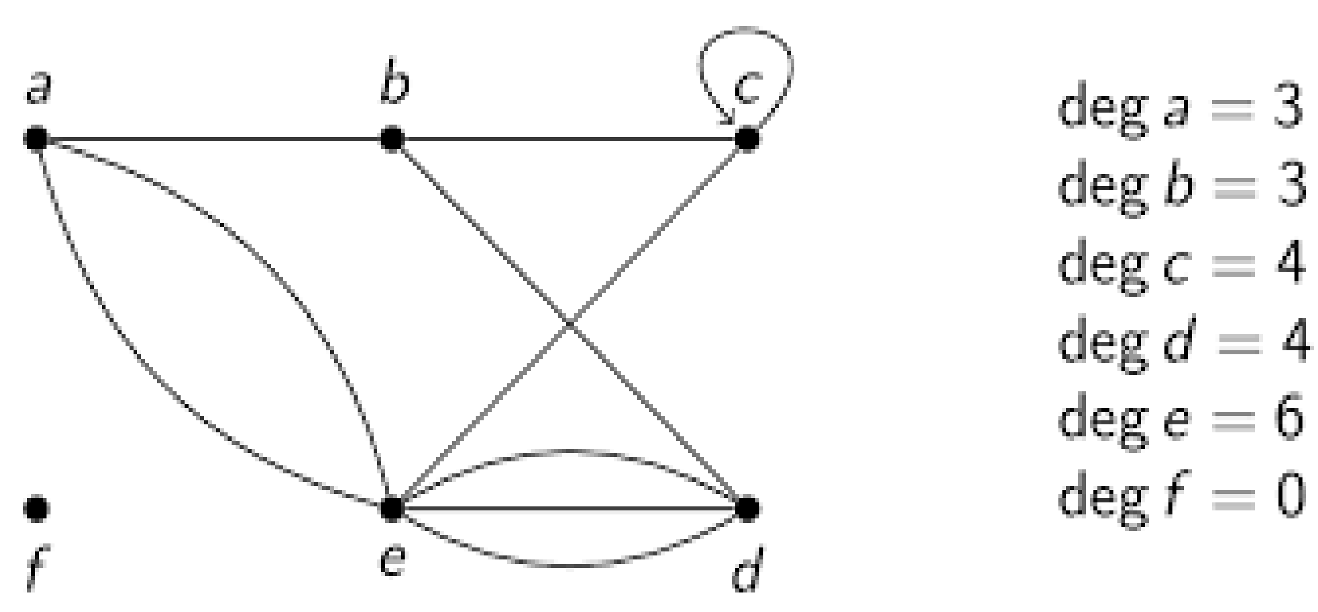
Observations -

- Total number of vertex = |V(G)| = n
- Total Number of edges = |E(G)| = e
- Odd Degree Vertex – a vertex incident with odd number of edges
- Even Degree vertex – a vertex incident with even number of edges





Degree-sum = 2 + 3 + 2 + 1 + 0 = 8 = 2*4 = 2e, n = 5, e= 4, odd degree vertex = 2, even degree vertex = 3

Degree-sum = 3 + 3 + 4 + 4 + 6 + 0 = 20 = 2*10 = 2e, n = 6, e = 10, odd degree vertex = 2, even degree vertex = 3

Degree – Sum Formula (Hand-Shaking Lemma)

If G is a graph, then $\sum_{v \epsilon V(G)} d(v) = 2e(G)$.

Proof – Summing the degrees counts each edge twice, since each edge has two ends and contributes to the degree at each endpoint.

- Corollary – Every graph has an even number of vertices of odd degree.

- Corollary – In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.

Thank You