Graph Theory : Some Important Graphs & Their Properties

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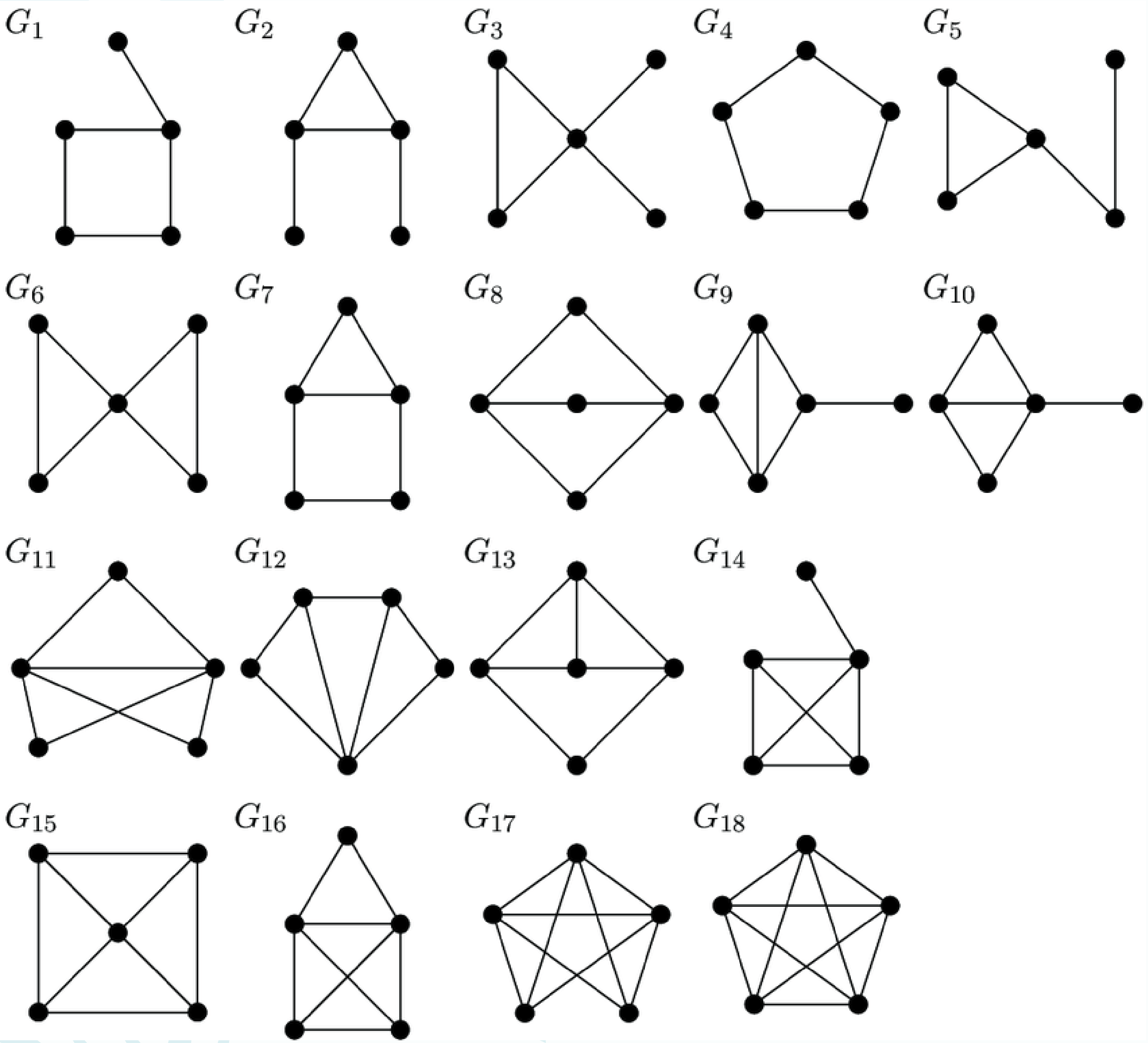
Recap -

- Definition of Graph
- Few important terms of Graph
- Degree of a vertex
- Degree – sum formula and its corollaries



Q. Can a simple graph have 5 vertices and 12 edges ?

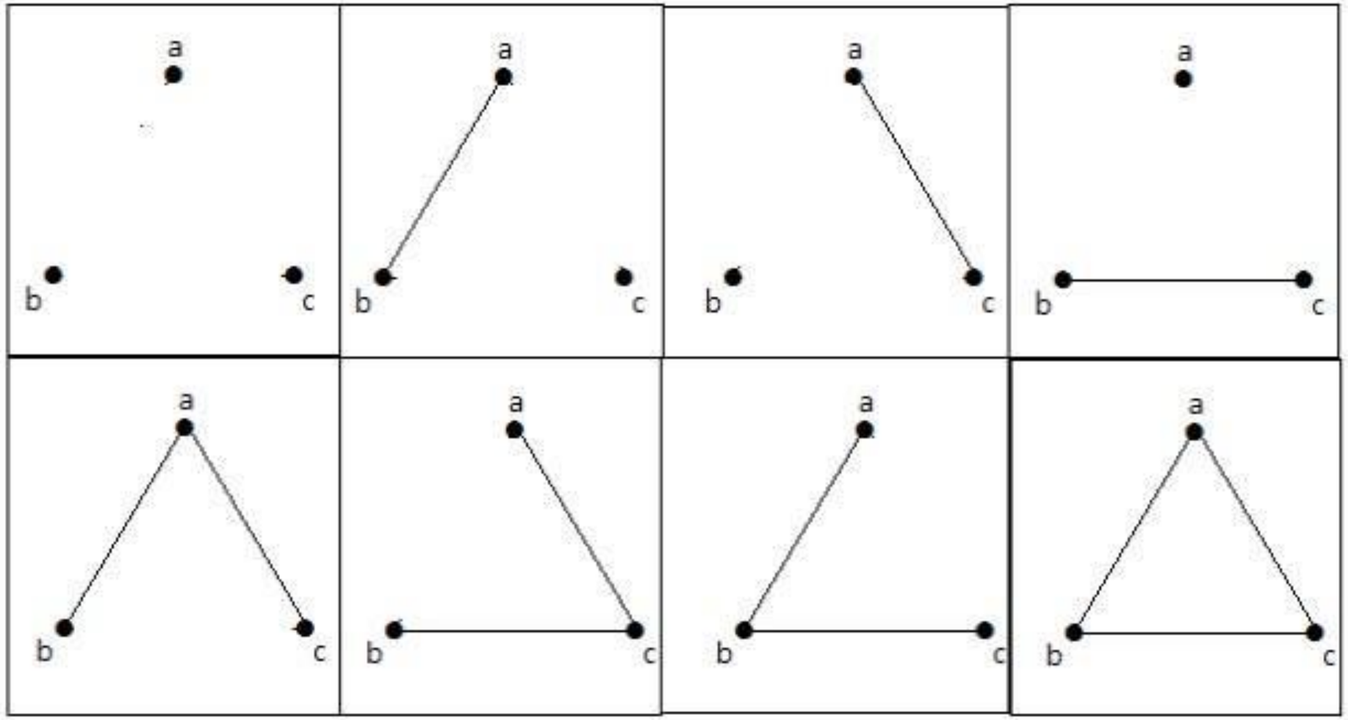
Ans – No..!!!

Observations – In the graph G_{18}

- Total number of edges are 10
- All the vertices are adjacent to each other
- Degree of every vertex is 4

- **The maximum number of edges possible in a simple graph with ‘n’ vertices is $^nC_2$ where $^nC_2 = n(n – 1)/2$.**

- **The number of simple graphs possible with ‘n’ vertices = $2^{^nC_2} = 2^{\frac{n(n-1)}{2}}$**

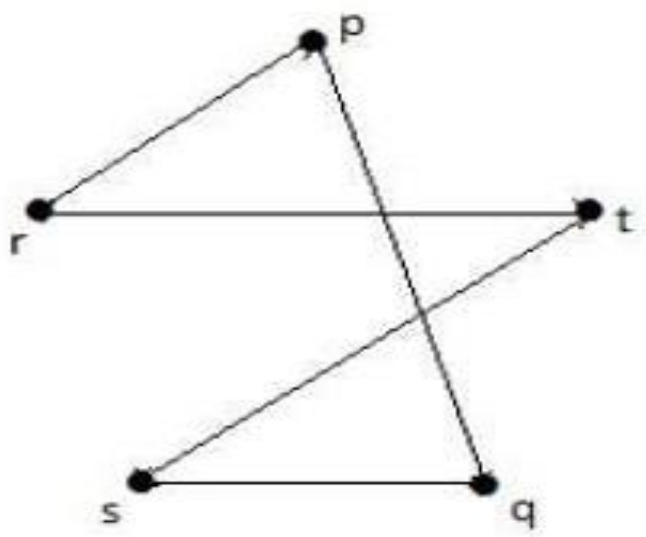


- The maximum number of edges with n=3 is
  $^nC_2 = n(n-1)/2 = 3(3-1)/2$
  $= 6/2 = 3$ edges

- The maximum number of simple graphs with n=3 is
  $2^{^nC_2} = 2^{n(n-1)/2} = 2^{3(3-1)/2} = 2^3 = 8$

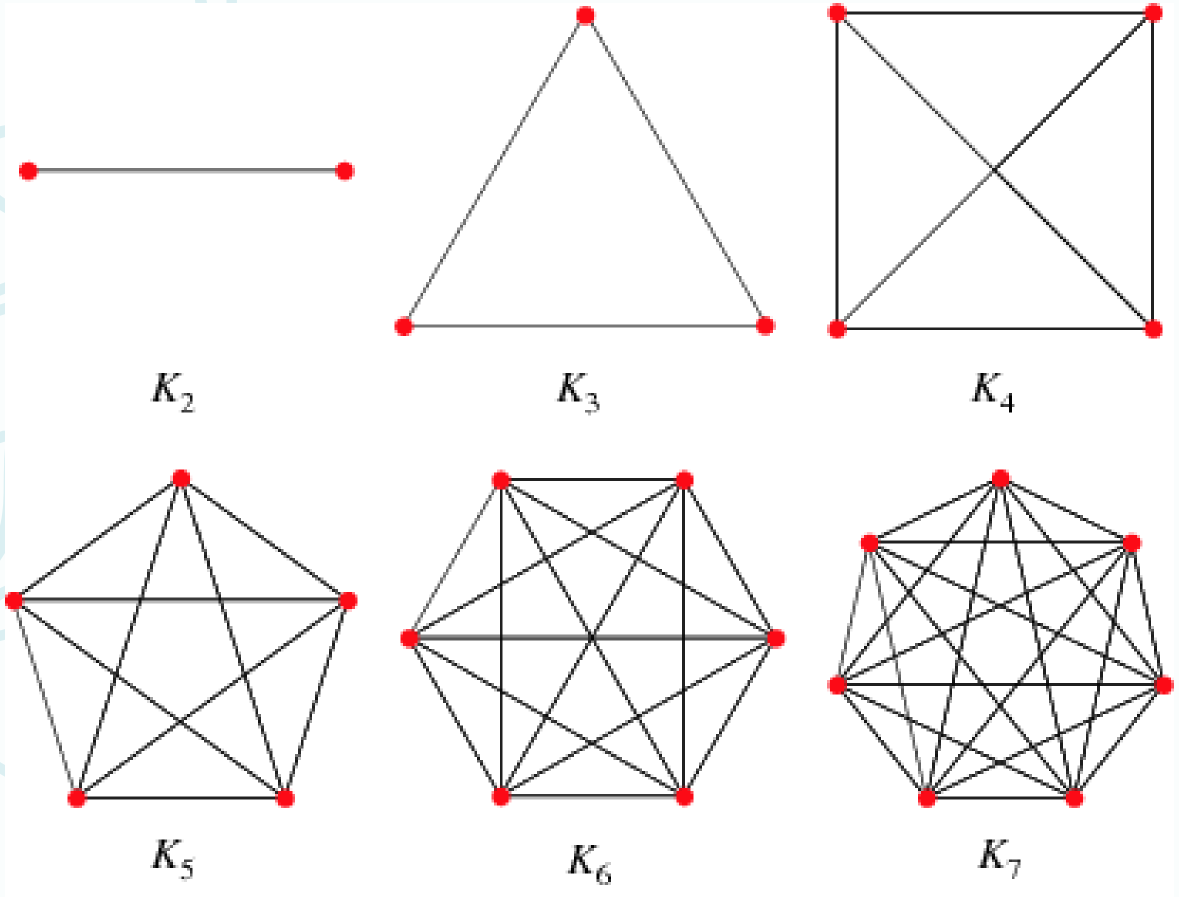
❑**Complete Graph -** A simple graph with n mutual vertices is called a complete graph and it is **denoted by C_n**. In the graph, **a vertex should have edges with all other vertices,** then it called a complete graph.

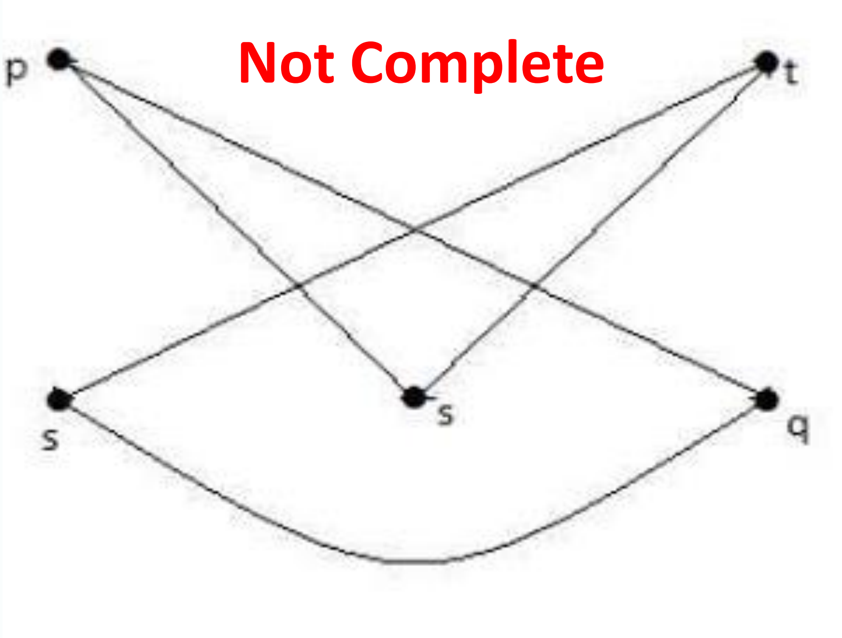
- In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.

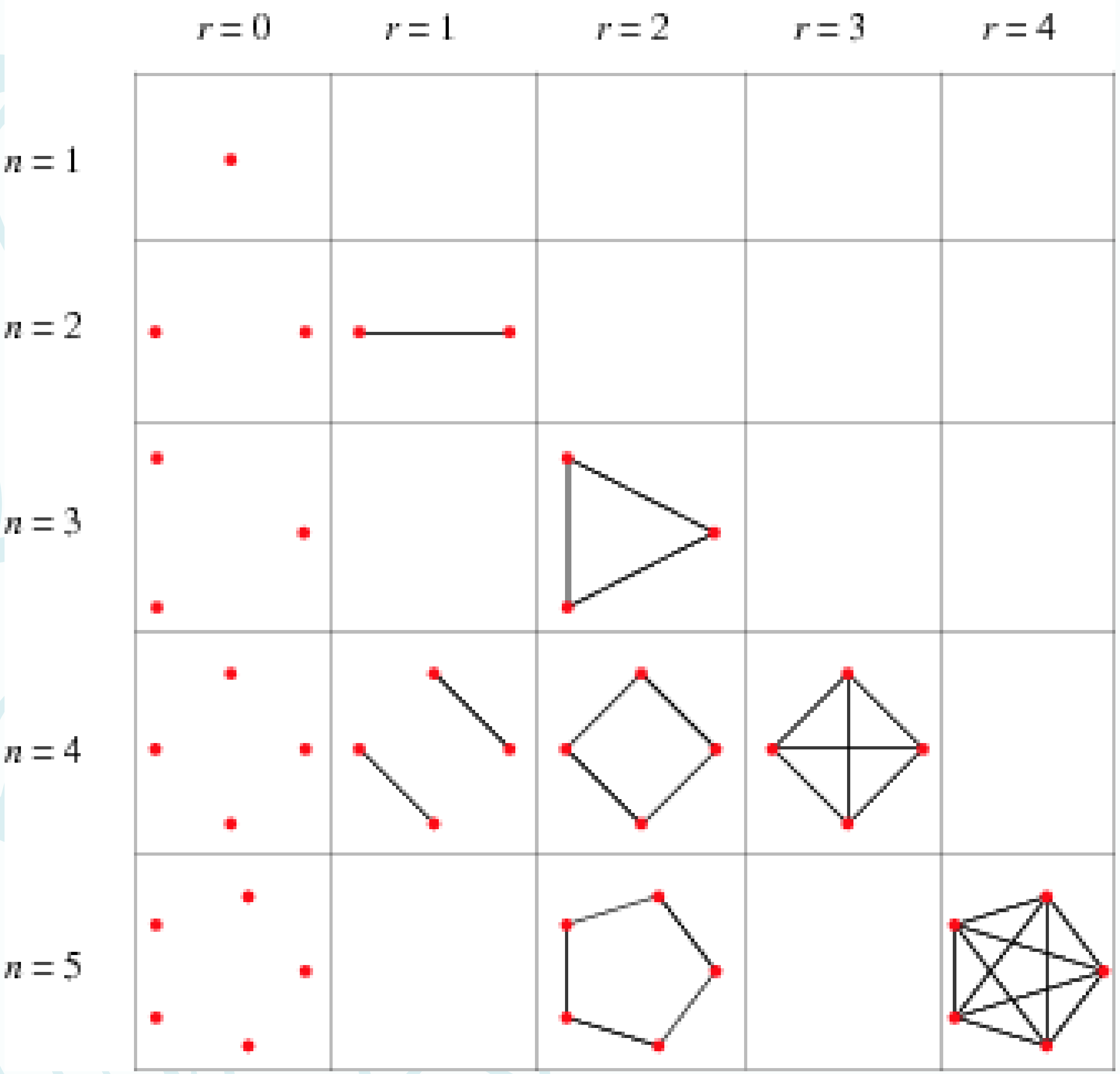


Not Complete

❑ Example: Complete Graph







- **Regular Graph** - a regular graph is a graph where each vertex has the same number of neighbors i.e. degree of every vertices are same.

- A graph is called **r - regular** if degree of each vertex in the graph is r.

❑ Note –

- Any (n-1) regular graph with n vertices is complete.
- A complete graph with n vertices is (n-1) regular.

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❑ Theorem - For a, r - Regular graph, if r is odd, then the number of vertices of the graph must be even.

Proof :

- Lets assume, number of vertices, n is odd.
- From Handshaking Theorem we know, sum of degree of all the vertices = 2 * no of edges of the graph …….(1)
- The R.H.S of the equation (1) is a even number.
- For a r - regular graph, each vertex is of degree r.
- Sum of degree of all the vertices = r * n, where r and n both are odd. So their product (sum of degree of all the vertices) must be odd.
- This makes L.H.S of the equation (1) is a odd number.
- So L.H.S *not equals* R.H.S.
- So our initial assumption that n is odd, was wrong. S
- Hence, number of vertices n must be even.

❑ Theorem - Number of edges of a r - Regular graph with n vertices = (n*r)/2.

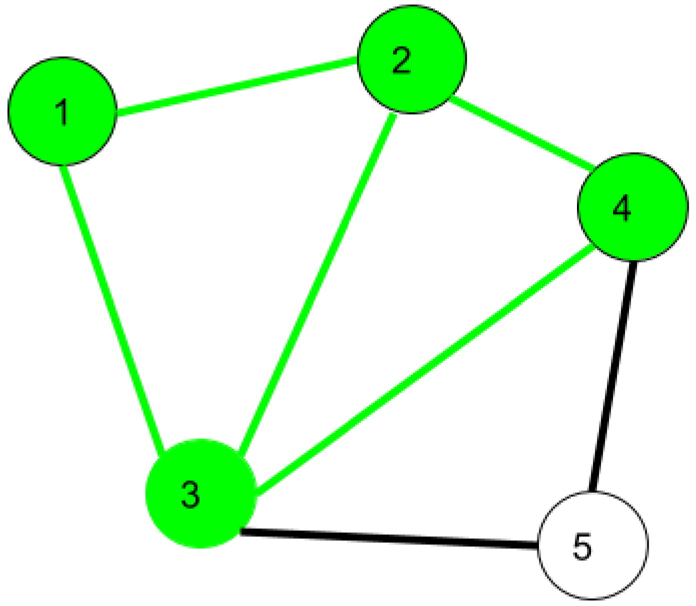
Proof:

- Let, the number of edges of a r - Regular graph with n vertices be e.
- From Handshaking Theorem we know, Sum of degree of all the vertices = 2 * e
- Hence n * r = 2 * e or, e = (n * r)/2

- **Walk** – A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk. Hence a walk is a list $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges such that $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. Vertex can be repeated. Edges can be repeated.

- **Open walk** - A walk is said to be an open walk if the starting and ending vertices are different i.e. the origin vertex and terminal vertex are different.

- **Closed walk** - A walk is said to be a closed walk if the starting and ending vertices are identical i.e. if a walk starts and ends at the same vertex, then it is said to be a closed walk.



Here 1->2->3->4->2->1->3 is a walk

- **In the given diagram:**
  1->2->3->4->5->3 is an open walk.
  1->2->3->4->5->3->1 is a closed walk.

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- **Trail –** A trail is a open walk with no repeated edges, but vertex can be repeated.

- A u-v walk or u-v trail has first vertex u and last vertex v; these are its endpoints.

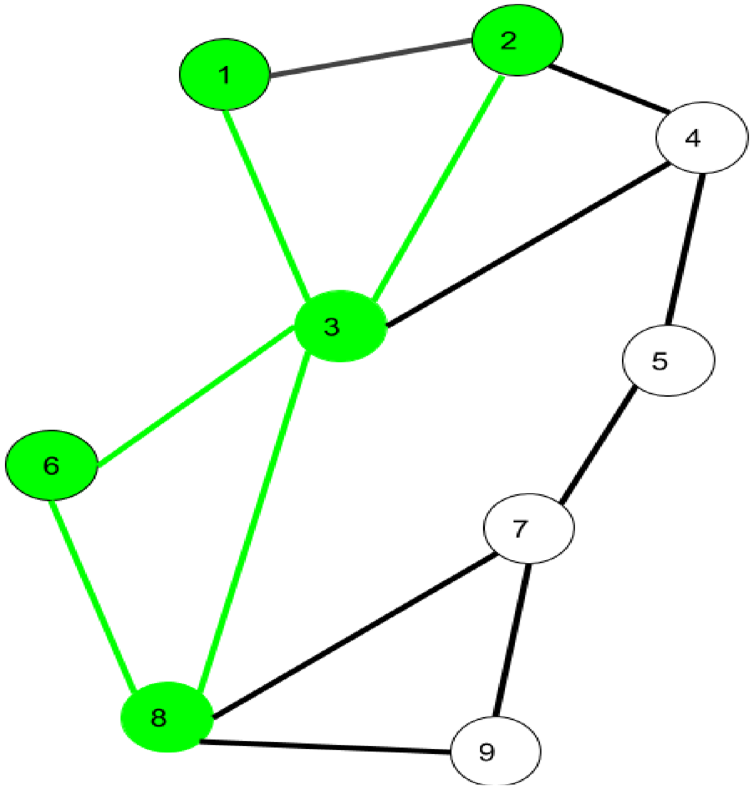
- **Circuit –** Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also i.e. it is a closed trail.

✓ Vertex can be repeated

✓ Edge not repeated

✓ Closed trail

- **Here 1->2->4->3->6->8->3->1 is a circuit**



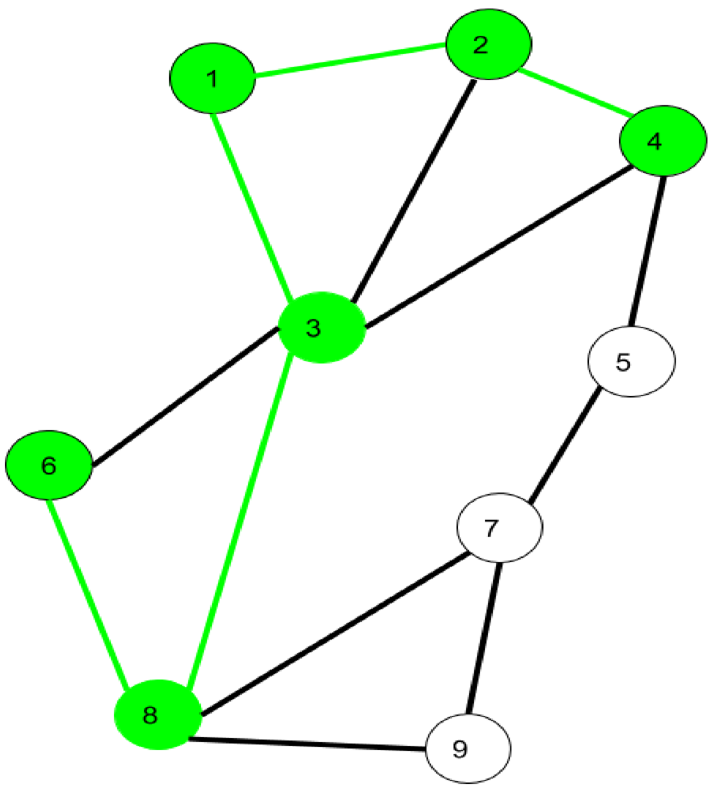
Here 1->3->8->6->3->2 is trail

❑ **Path** – It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge.

- As path is also a trail, thus it is also an open walk.

✓ Vertex not repeated
✓ Edge not repeated

➢ <u>The length of a walk, trail, path or cycle is its number of edges.</u>

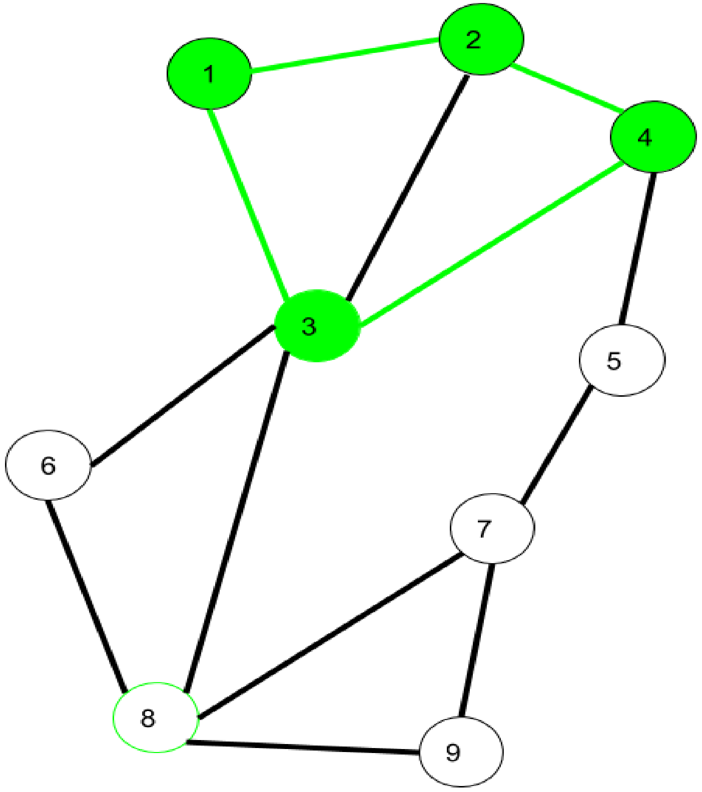


Here 6->8->3->1->2->4 is a Path

- **Cycle** – Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a cycle.

✓ Vertex not repeated
✓ Edge not repeated
✓ Closed Path

- **Note that for closed sequences start and end vertices are the only ones that can repeat.**



Here 1->2->4->3->1 is a cycle

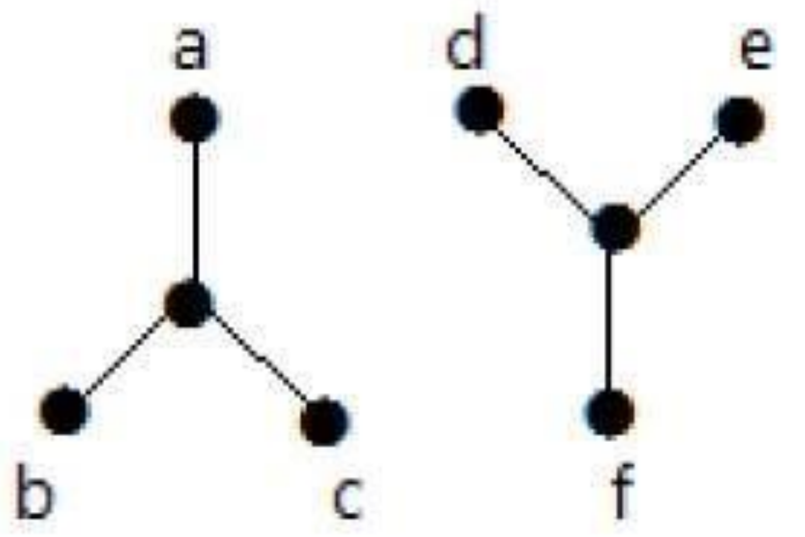
❑ Results -

1. When a graph G contains a u-v walk of length k, then G has a u-v path at most of length k.

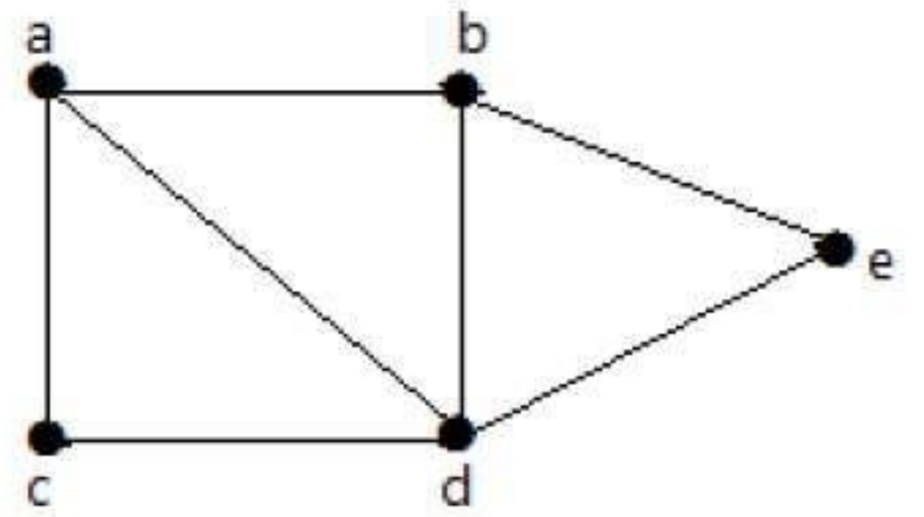
2. Every vertex, except the terminal vertices in a u-v walk, whose all edges are distinct (i.e. in a u-v trail) is an even degree vertex.

3. Every u-v trail contains a u-v path.

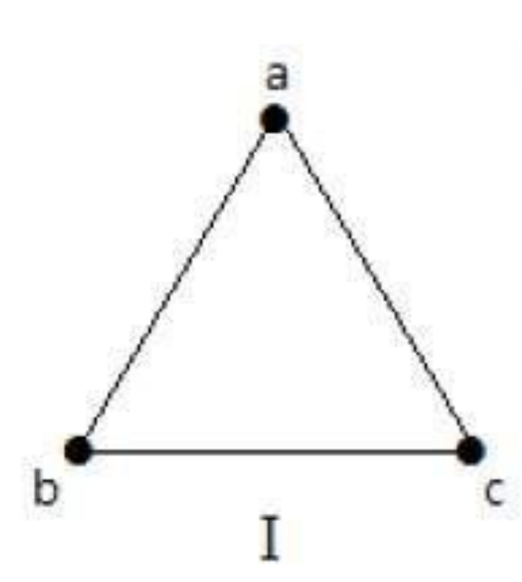
- **Connectivity -** A graph is said to be **connected if there is a path between every pair of vertex.** From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

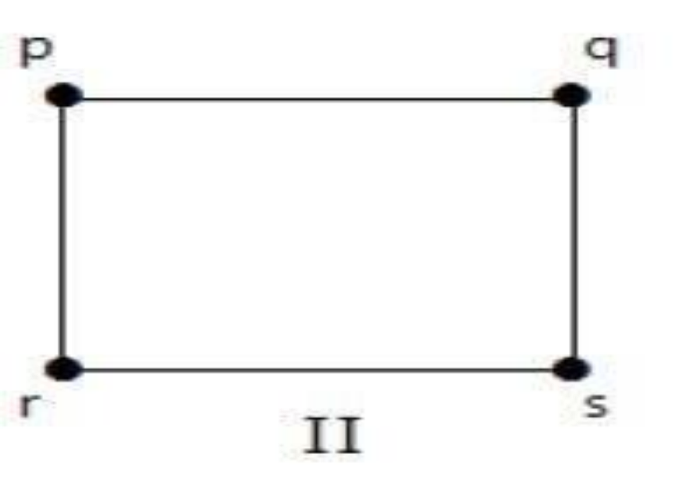


traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.



it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'



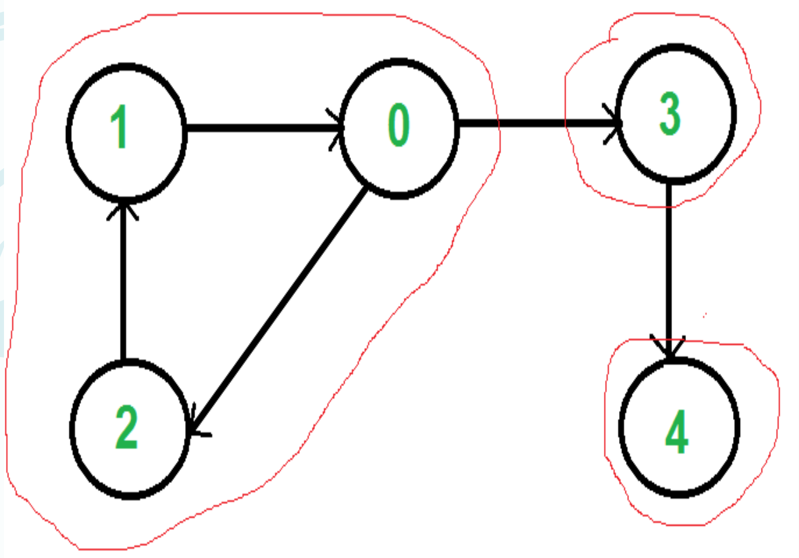


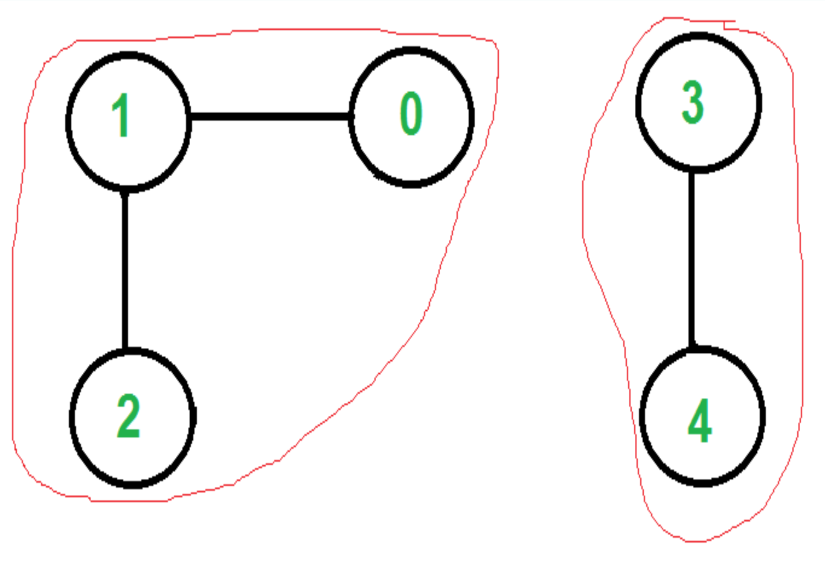
Graph I	a	b	c
a	Not Connected	Connected	Connected
b	Connected	Not Connected	Connected
c	Connected	Connected	Not Connected

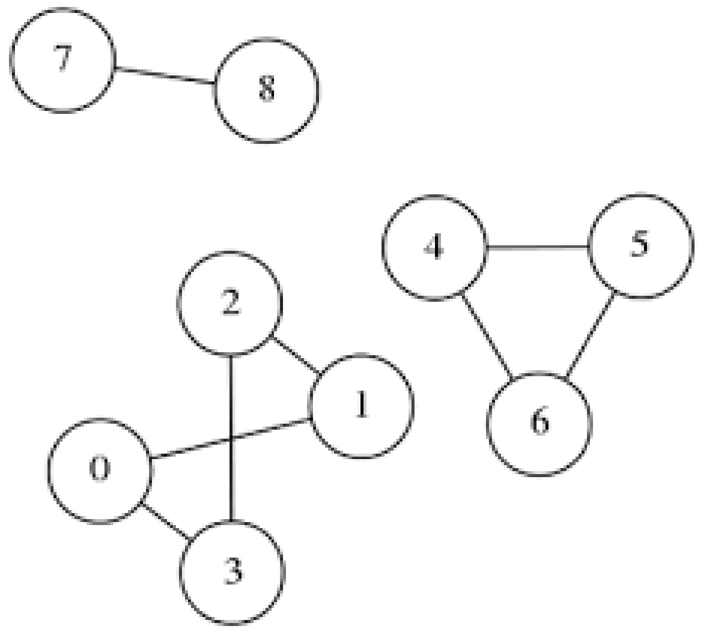
Graph II	p	q	r	s
p	Not Connected	Connected	Connected	Connected
q	Connected	Not Connected	Connected	Connected
r	Connected	Connected	Not Connected	Connected
s	Connected	Connected	Connected	Not Connected

❑Components of Graph

In a graph G, a connected subgraph of it will be called the component of G if it is not contained in any bigger connected subgraph of G. In the case of a disconnected graph it may consists of two or more connected subgraphs. Each of these connected subgraphs is known as component of the graph







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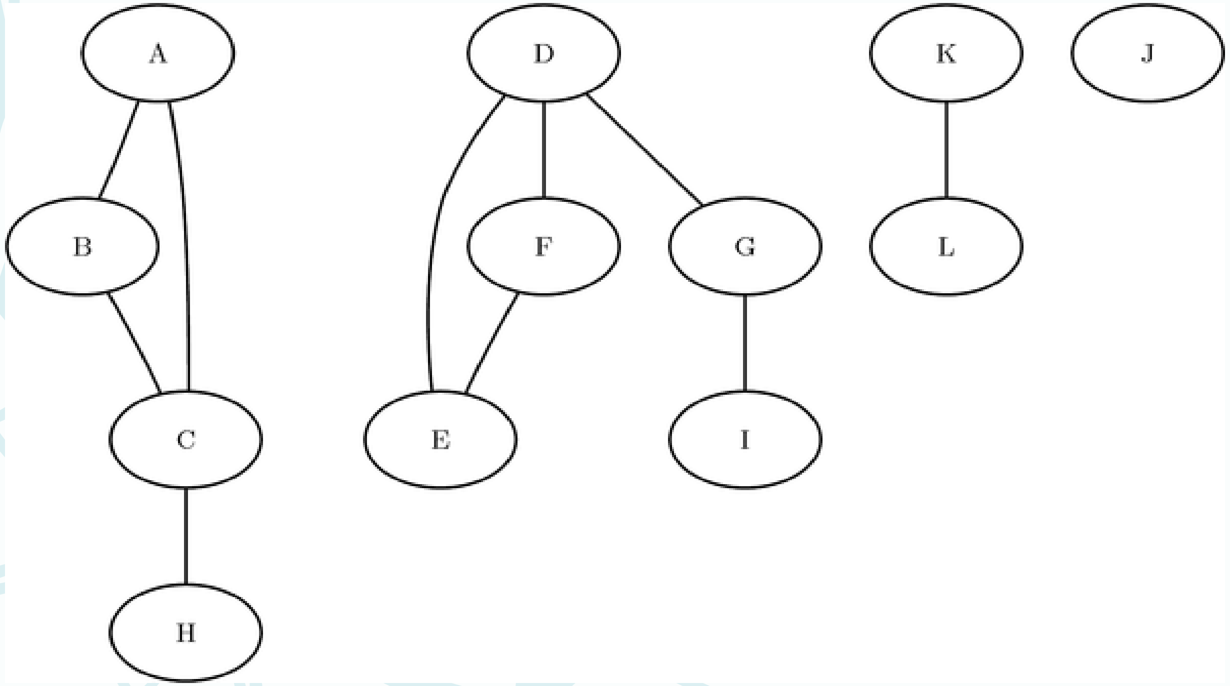


❑ Results -

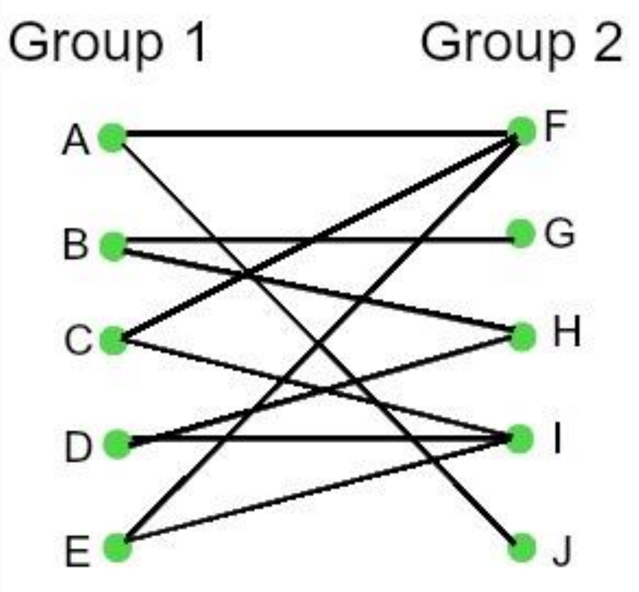
- The minimum number of edges in a connected graph with n vertices is n-1.
- The maximum number of edges in a connected simple graph with n vertices is $\frac{n(n-1)}{2}$.
- The minimum number of edges in a graph with n vertices and k components is n-k.
- A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$.

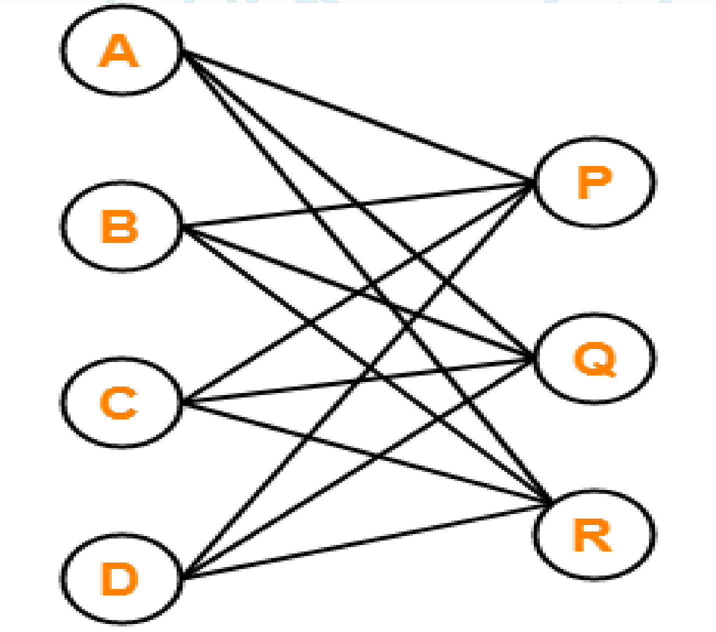
Total number of vertices – 12
Total number of edges – 10
Total number of components - 4
Minimum edges – 12 – 4 = 8
Maximum edges - $\frac{(12-4)(12-4+1)}{2} = \frac{8*9}{2} = 36$

Note – This is not the ONLY graph having 12 vertices and 4 connected components.

❑ Bipartite Graph

If the vertex-set of a graph G can be split into two disjoint sets, V_1 and V_2 , in such a way that each edge in the graph joins a vertex in V_1 to a vertex in V_2 , and there are no edges in G that connect two vertices in V_1 or two vertices in V_2 , then the graph G is called a bipartite graph.





❑ Complete Bipartite Graph

A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to every single vertex in the second set. The complete bipartite graph is denoted by $K_{x,y}$ where the graph G contains x vertices in the first set and y vertices in the second set.

❑ Result - A bipartite graph with n vertices has at most $\frac{n^2}{4}$ edges.

Proof –

- Let the n vertices in a bipartite graph , the vertex set A consists of m vertices and the vertex set B consists of (n-m) vertices.
- Since each vertex of set A is connected with each vertex of set B, so total number edges in this bipartite graph is m*(n – m) = f(m), say.
- Now to get maximum values of f(m), we get

$$f'(m) = 0,$$
$$\text{i.e. } (n - m) + m(-1)=0,$$
$$\text{i.e. } n - 2m=0,$$
$$\text{i.e. } m = \frac{n}{2}$$

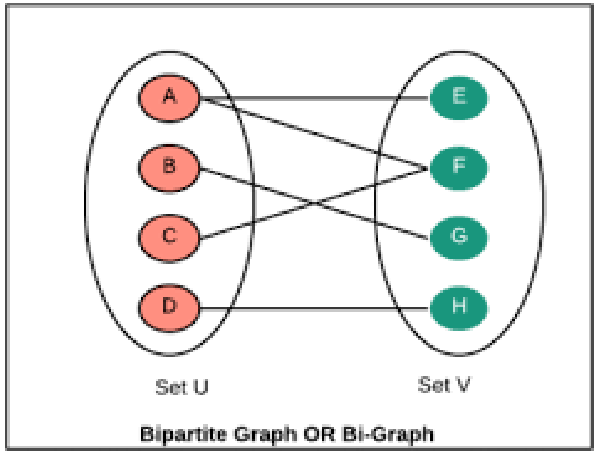
$$Again, f''(m) = -2 < 0$$

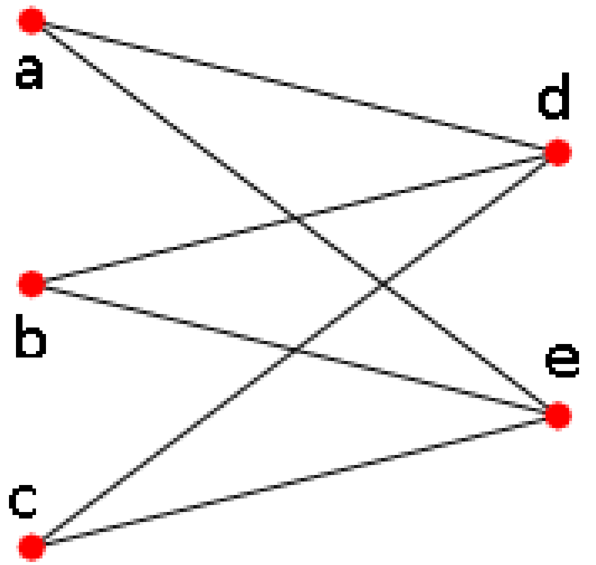
$$so, f(m) is\ maximum\ for\ m = \frac{n}{2}$$

$$Hence, f_{max}(m) = \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{n^2}{4}$$

$$Thus\ the\ maximum\ number\ of\ edges\ is \frac{n^2}{4}$$

❑ Result - A bipartite graph cannot contain a cycle of odd length.





Proof –

- Let the given graph has a cycle of length n.
- Also, let V = V_1 U V2. Now consider the cycle as - let v_1 and v_2 are joined by the edge e_1, v_2 and v_3 joined by the edge e_2….. and finally v_{n-1} and v_n are joined by e_{n-1} and since this graph is a cycle so v_n and v_1 is joined by the edge e_n.
- Let the cycle C be denoted by $v_1e_1v_2e_2v_3….v_{r-1}e_{r-1}v_r……v_{n-1}e_{n-1}v_ne_nv_1$ and for this cycle , n edges are required.
- Now it is clear that, since $v_n \in V_2$ so n is even.
- Thus this cycle is composed of even number of edges.
- Hence, Theorem is proved.

➢ **Converse of the above is also true.**

If a simple graph contains no cycle of odd length, then this graph is a bipartite graph.

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Thank You