Graph Theory : Matrix Representation of Graphs

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Recap -

- Complete Graph
- Regular Graph
- Walk, Path, Circuit
- Bi-partite Graph

❑Matrix Representation of a Graph

There are so many matrix representations of any graph, of which two are very important.

- Incidence Matrix Representation
- Adjacency Matrix Representation

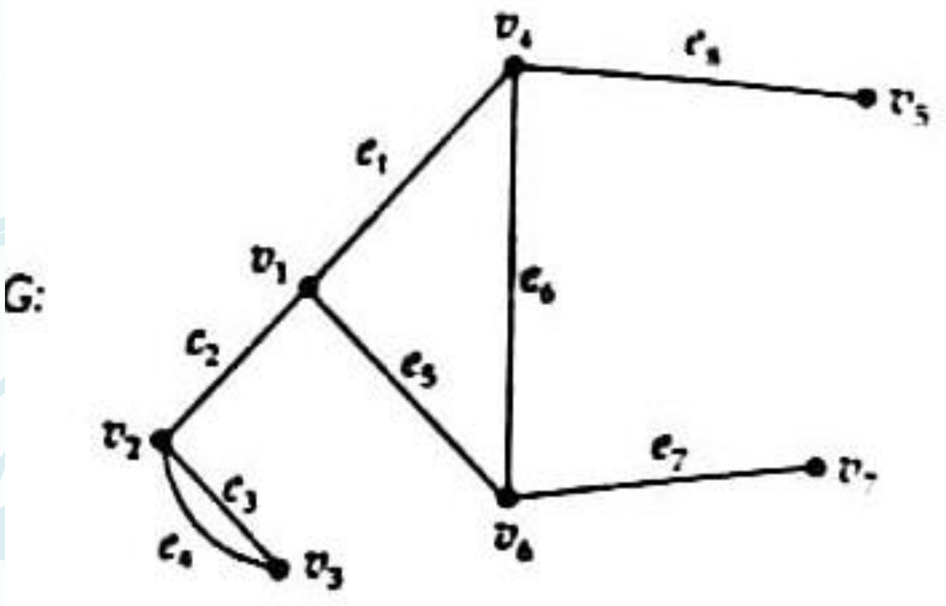
❑Incidence Matrix Representation of a Graph

- Let G be a graph with n vertices and m edges and let G contains no self loop. Now we define an n x m matrix I(G) given by I = $(a_{ij})_{nxm}$, where rows correspond vertices and columns correspond to the edges and aij's are selected as follows

$$a_{ij} = \begin{cases} 1, if\ j-th\ edge\ e_j\ incidents\ on\ the\ i-th\ vertex\ v_i \\ 0, otherwise \end{cases}$$

Note : In incidence matrix, every column will contain only two 1's.

❑ Example – Incidence Matrix



$I(G) =$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	0	0	1	0	0	0
v_2	0	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	0	0
v_4	1	0	0	0	0	1	0	1
v_5	0	0	0	0	0	0	0	1
v_6	0	0	0	0	1	1	1	0
v_7	0	0	0	0	0	0	1	0

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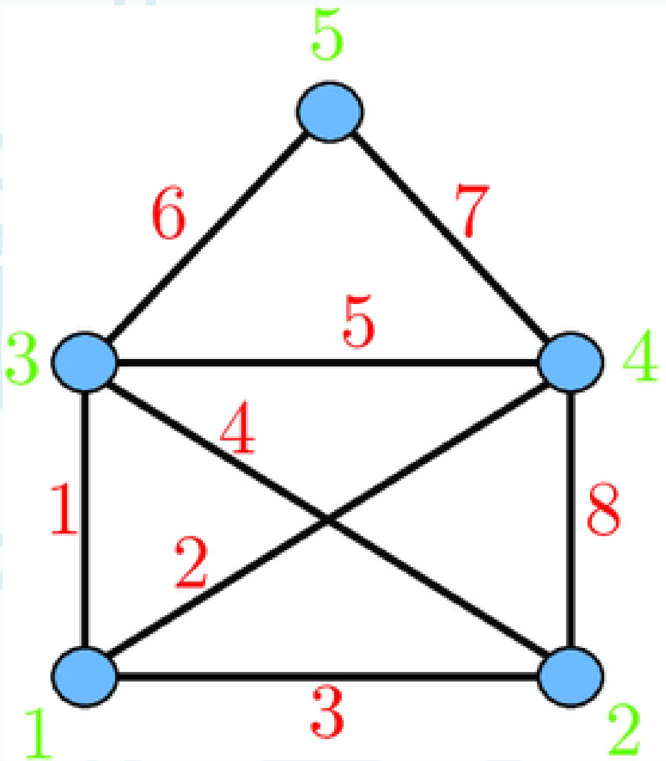
❑ Example – Incidence Matrix



$\mathbf{A}_G =$

	1	2	3	4	5	6	7	8
1	1	1	1	0	0	0	0	0
2	0	0	1	1	0	0	0	1
3	1	0	0	1	1	1	0	0
4	0	1	0	0	1	0	1	1
5	0	0	0	0	0	1	1	0

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❑Adjacency Matrix of a Graph

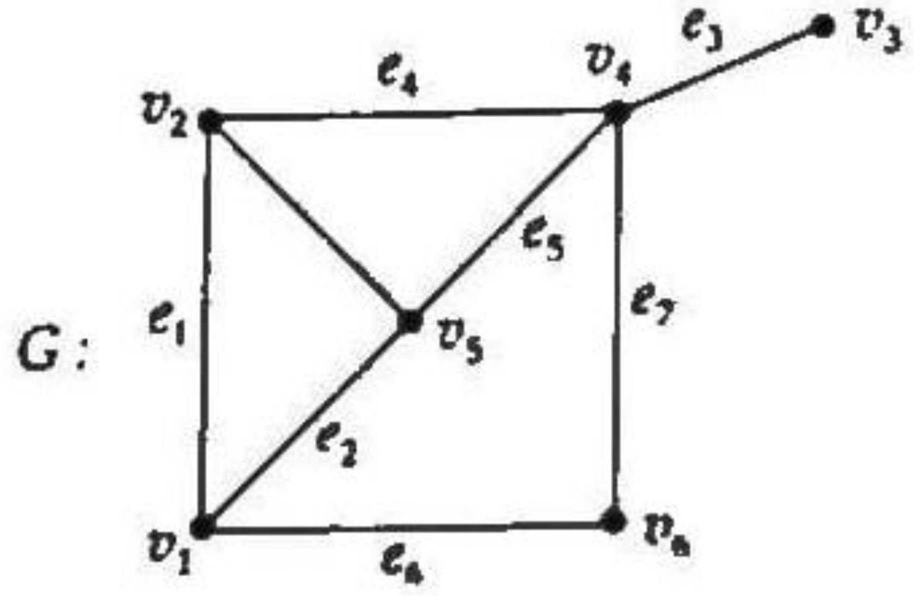
For a connected graph G, having no parallel edges(but there may be loops). Then the adjacency matrix A of G is defined as $A(G) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} 1, when\ i-th\ and\ j-th\ vertices\ are\ connected\ by\ an\ edge \\ 0, when\ there\ is\ no\ edge\ between\ i-th\ and\ j-th\ vertex \end{cases}$$

Note : For a loop at the vertex v_i in a connected graph G, $a_{ii} = 1$.

Remark – In the case of simple graph G, i.e. having no loops and no parallel edges, the adjacency matrix of G is an n x n symmetric square matrix having all its diagonal elements '0' and $a_{ij} = a_{ji}$.

❑ Example – Adjacency Matrix



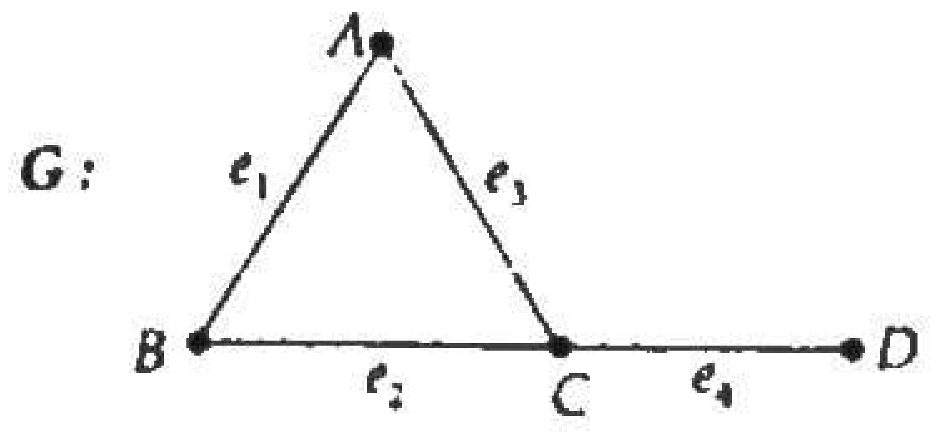
	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	0	0	1	1
v_2	1	0	0	1	1	0
v_3	0	0	0	1	0	0
v_4	0	1	1	0	1	1
v_5	1	1	0	1	0	0
v_6	1	0	0	1	0	0

- ✓ In a simple connected matrix for any vertex the entries in row or column of v_i, the number of 1's is equal to the degree of the vertex.
- ✓ For any square matrix with entries '0' and '1' and in which all the diagonal elements are '0', there exists a simple connected graph, corresponding to this matrix.

❑ Relation between Incidence matrix and Adjacency Matrix

- In case of a graph which contains no loop, its incidence matrix I(G) gives all the information about G.
- In case of a graph which contains no parallel edges, its adjacency matrix A(G) gives all the information about G.

- Therefore it is expected that in the case of simple graph G, one matrix can be obtained directly from the other.



A(G) =

	A	B	C	D
A	0	1	1	0
B	1	0	1	0
C	1	1	0	1
D	0	0	1	0

I(G) =

	e_1	e_2	e_3	e_4
A	1	0	1	0
B	1	1	0	0
C	0	1	1	0
D	0	0	0	1

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❑Adjacency Matrix of a Disconnected Graph -

If G be a disconnected graph with components G_1 and G_2, then the adjacency matrix of G is given by a block diagram form as $\begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$ where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 respectively, and [0] is a null matrix.

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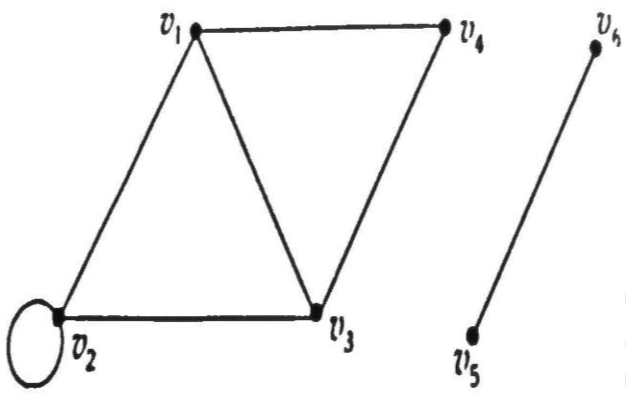
❑Example - Adjacency Matrix of a Disconnected Graph



$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

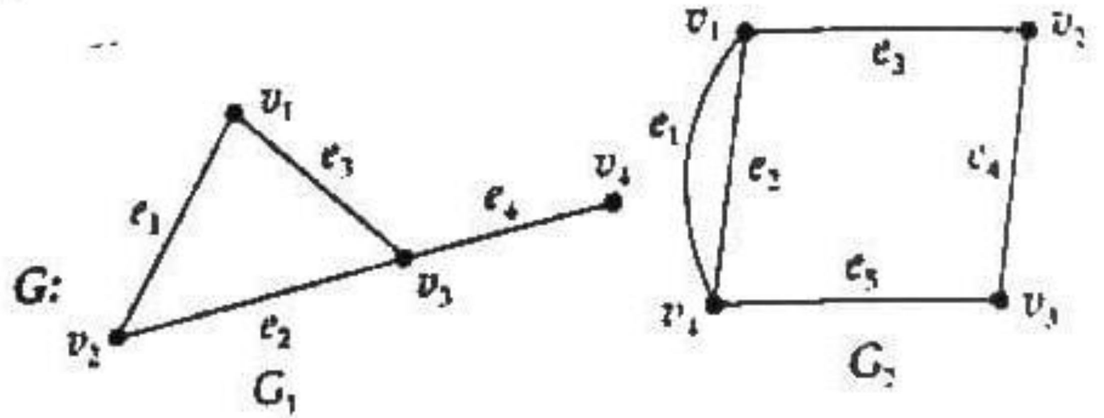
$$A(G_2) = \begin{array}{c} \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cc} v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$A(G) = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}, \text{ i.e.}$$

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

❑Incidence Matrix of Disconnected Graph -

If G be a disconnected graph with components G_1 and G_2, then the incidence matrix of G is given by a block diagram form as $\begin{bmatrix} I(G_1) & 0 \\ 0 & I(G_2) \end{bmatrix}$ where $I(G_1)$ and $I(G_2)$ are the incidence matrices of G_1 and G_2 respectively, and [0] is a null matrix.



$$A(G_2) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_1 & e_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

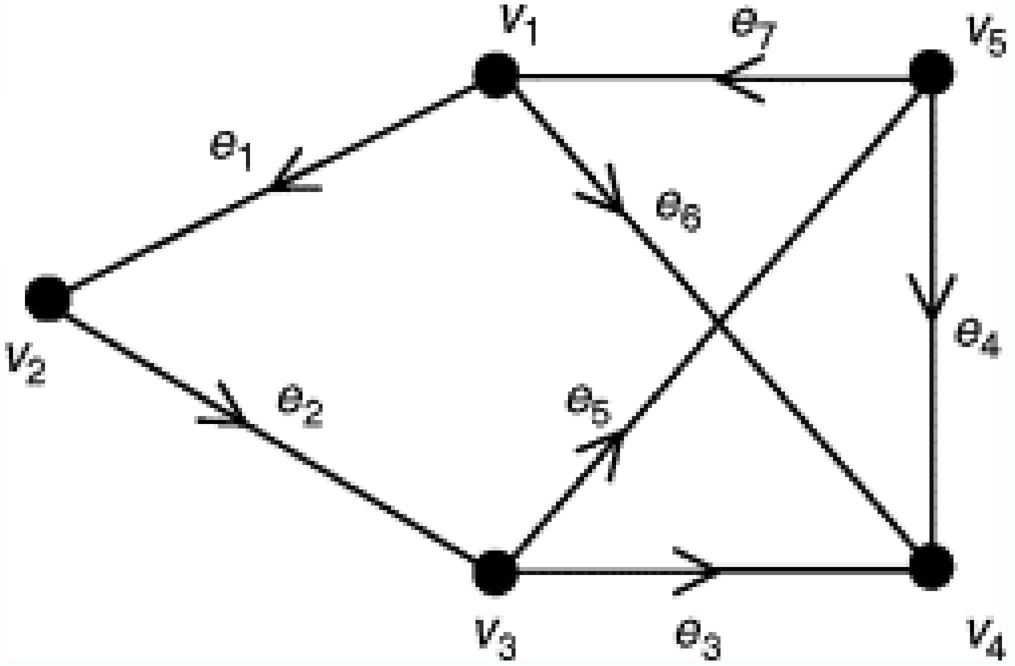
$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

Then, $$A(G) = \left[\begin{array}{cccc|ccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}.$$

❑ Di-Graph or Directed Graph

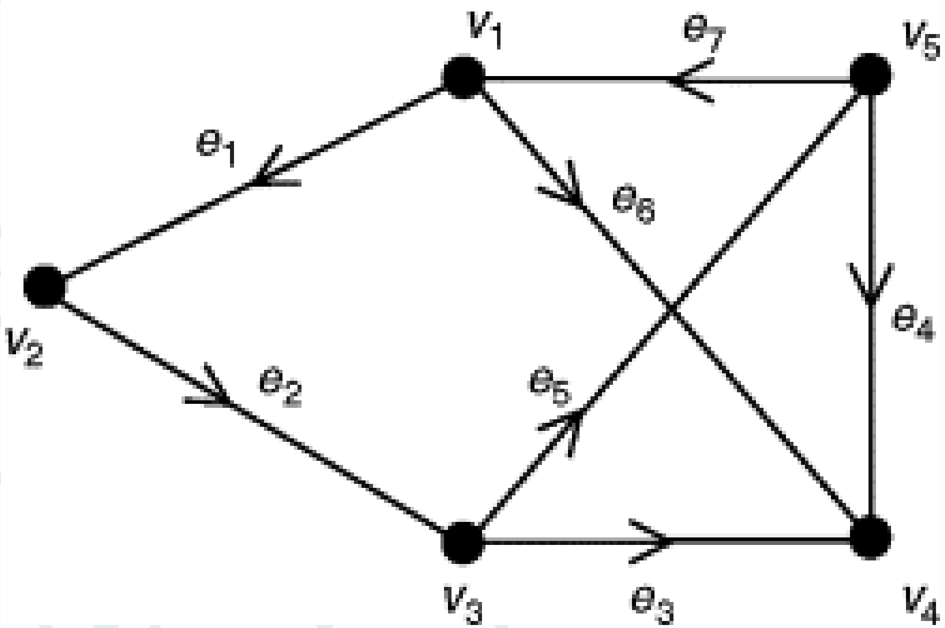
- A digraph or directed graph is a graph in which each edge e joining the vertices v_i and v_j has a definite direction from its initial vertex v_i to its terminal vertex v_j.

✓ A graph having no direction is called an Undirected graph.



❑ Out degree and In degree of a vertex in a digraph

- The number of edges leaving any vertex is called its out degree
- The number of edges entering in any vertex is called its in degree.

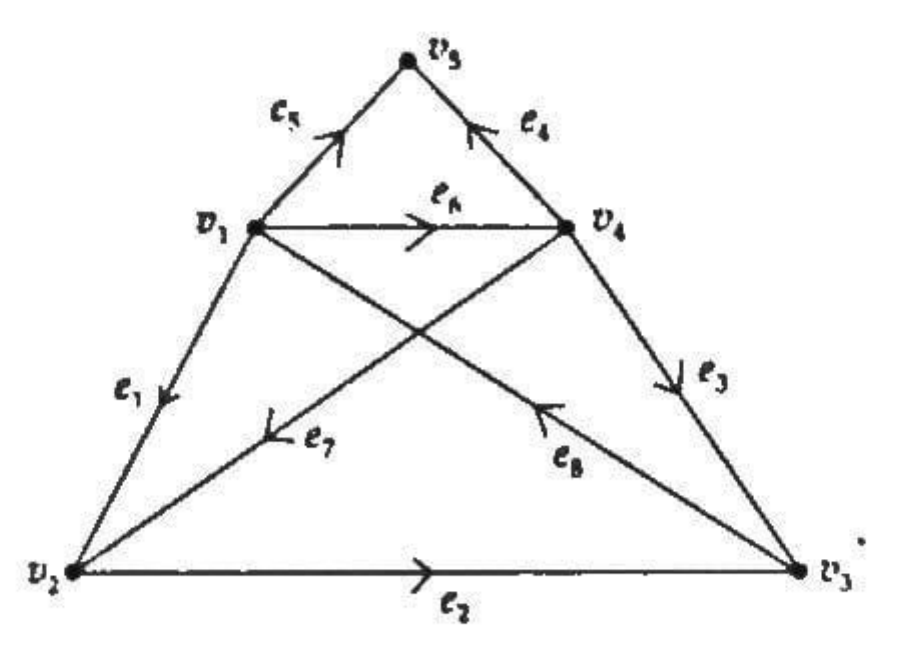


	Out degree	In degree	Degree sum
V_1	2	1	3
V_2	1	1	3
V_3	2	1	3
V_4	0	3	3
V_5	2	1	3

❑ Incidence Matrix of a Connected Digraph

Let G be connected digraph with n vertices and m edges; let G contains no self-loop. Now we define incidence matrix $I(G) = (a_{ij})_{n\times m}$ as follows

$$a_{ij} = \begin{cases} 1, & if\ j-th\ edge\ is\ incident\ out(coming\ out)\ of\ the\ i-th\ vertex \\ -1, & if\ j-th\ edge\ is\ incident\ into(coming\ into) the\ i-th\ vertex \\ 0, if\ j-th\ edge\ is\ neither\ incident\ out\ nor\ incident\ into\ the\ i-th\ vertex \end{cases}$$



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	0	0	0	1	1	0	–1
v_2	–1	1	0	0	0	0	–1	0
v_3	0	–1	–1	0	0	0	0	1
v_4	0	0	1	1	0	–1	1	0
v_5	0	0	0	–1	–1	0	0	0

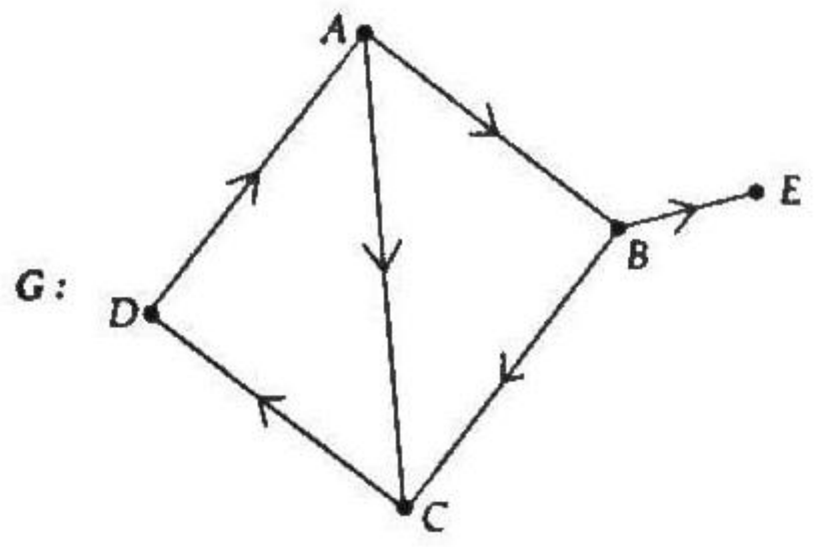
❑ Adjacency Matrix of a Digraph -

In the case of a digraph G, having no parallel edges (but there may be self loop), if v_1, v_2, v_3,…, v_n be the vertices, then its adjacency matrix A(G) is defined as $A = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} 1, when\ there\ is\ an\ edge\ directed\ from\ v_i\ to\ v_j \\ 0, \qquad if\ there\ is\ no\ edge\ between\ v_i\ and\ v_j \end{cases}$$

Note –

- ✓ A self – loop at the vertex v_i the corresponding entry $a_{ii} = 1$
- ✓ For digraph adjacency matrix A(G) is not symmetric.



$$A(G) = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

- ✓ If the adjacency matrix A(G) of any graph G is not symmetric, then for this A(G), there corresponds a digraph.
- ✓ The number of 1's in any row represents the number of out degree of the corresponding vertex.
- ✓ The number of 1's in ay column represents the number of in degree of the corresponding vertex.

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Thank You