Graph Theory : Hamiltonian & Eulerian Graph

Prepared by – Subhra Sarkar
Assistant Professor, BSH Dept.
Bengal Institute of Technology

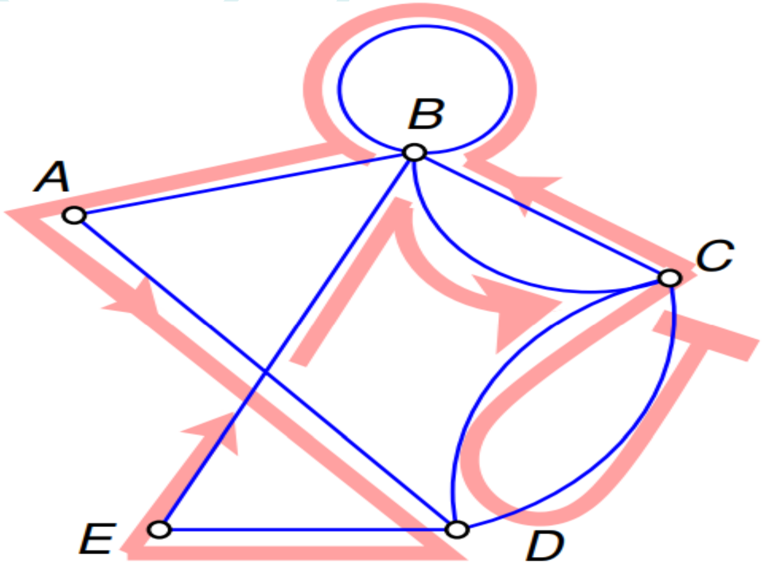
Recap

- Matrix Representation of Graph
- Incidence Matrix
- Adjacency Matrix

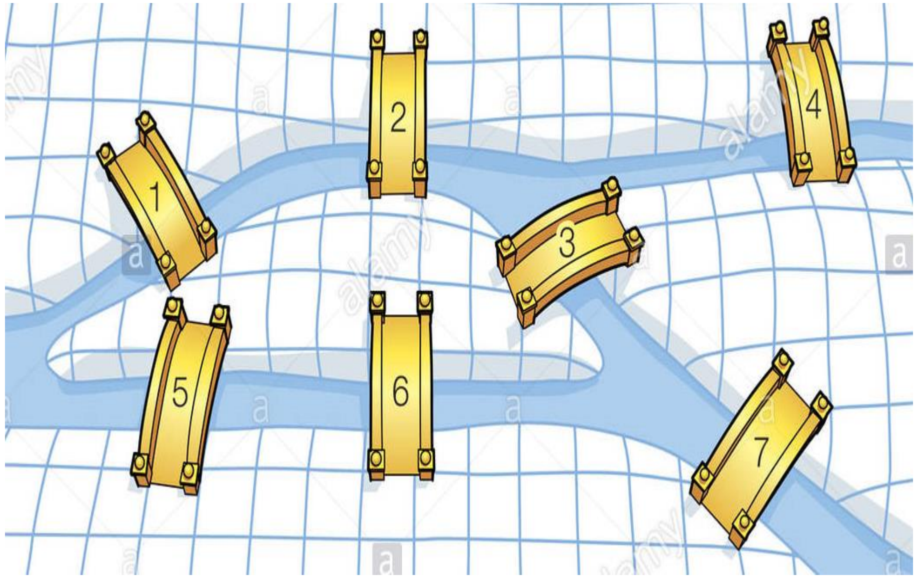
- Fundamentals of Graph
- Complete Graph
- Regular Graph
- Bi-partite Graph
- Complete Bipartite Graph

- Certain graph problems deal with finding a path between two vertices such that ***each edge is traversed exactly once, or finding a path between two vertices while visiting each vertex exactly once***. These paths are better known as **Euler path** and **Hamiltonian path** respectively.

- Chinese Postman Problem or Route Inspection Problem
- Traveling Salesman Problem
- *Konigsberg Bridge Problem*



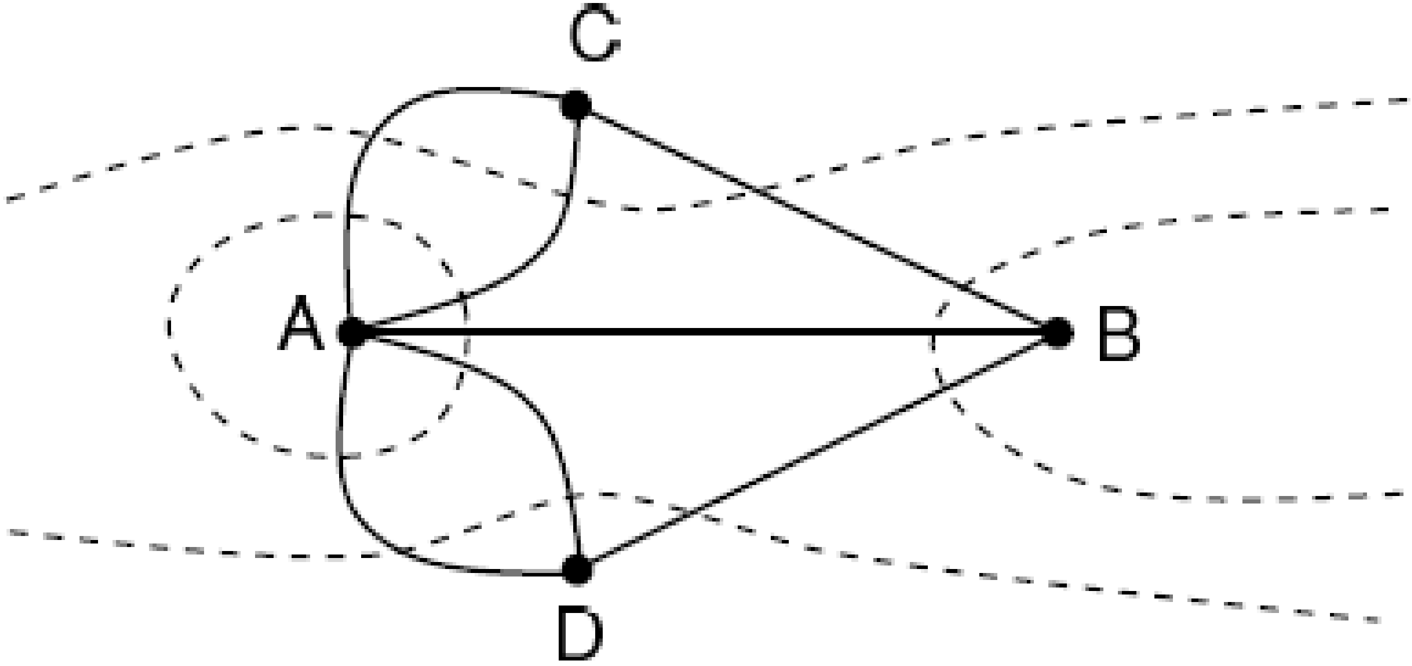
An Euler circuit: CDCBBADEBC



❑ Eulerian Graph -

- Problem - The town of Konigsberg consists of two islands and seven bridges. Is it possible, by beginning anywhere and ending anywhere, to walk through the town by crossing all seven bridges but not crossing any bridge twice?

- Solution - It is in fact impossible to walk through the town and traverse all the bridges only once.

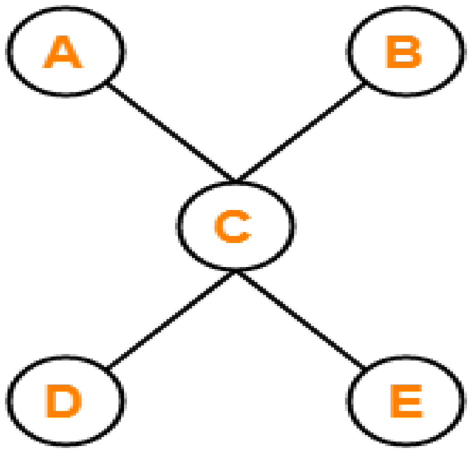


The bridges of Königsberg problem

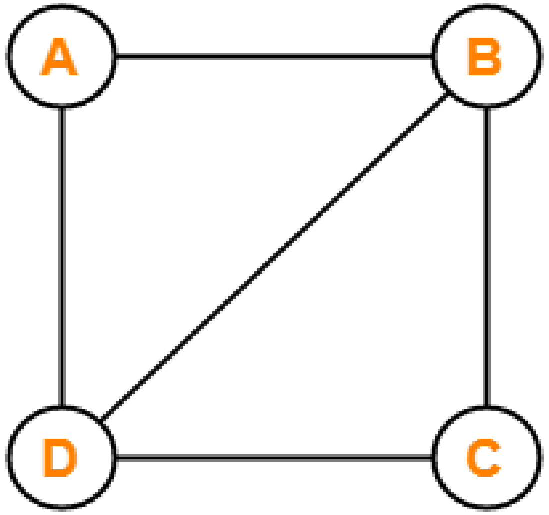
- Eulerian trail: An Eulerian trail is a trail that visits every edge of the graph once and only once. It can end on a vertex different from the one on which it began.

- Eulerian Circuit: An Eulerian circuit is an Eulerian trail that is a circuit. That is, it begins and ends on the same vertex.

- Eulerian Graph: A graph is called Eulerian when it contains an Eulerian circuit.



Euler Circuit Does Not Exist



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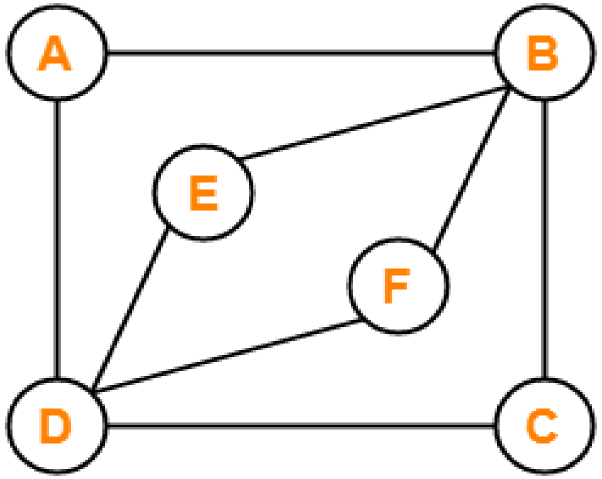

Euler Circuit = ABCDFBEDA

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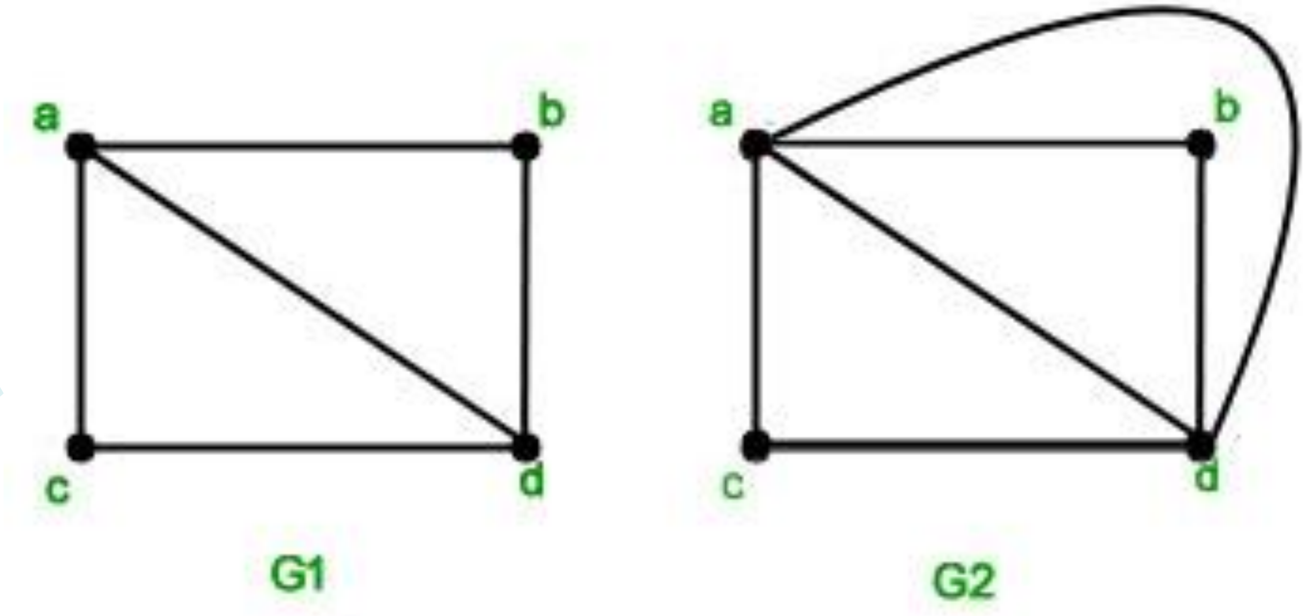
Result - An Eulerian trail exists in a connected graph if and only if there are either no odd vertices or two odd vertices.

Result - A connected multigraph (and simple graph) with at least two vertices has a Euler circuit if and only if **each** of its vertices has an **even degree**.

- ***Note: For the case of no odd vertices, the path can begin at any vertex and will end there. For the case of two odd vertices, the path must begin at one odd vertex and end at the other. A traversable trail may begin at either odd vertex and will end at the other odd vertex. Since a path may start and end at different vertices, the vertices where the path starts and ends are allowed to have odd degrees.***

- ***From this we can see that it is not possible to solve the bridges of Konigsberg problem because there exists within the graph more than 2 vertices of odd degree.***



- G_1 has two vertices of odd degree, a and d, the rest of them have even degree. So this graph has an Euler path but not an Euler circuit. The path starts and ends at the vertices of odd degree. The path is a-c-d-a-b-d.

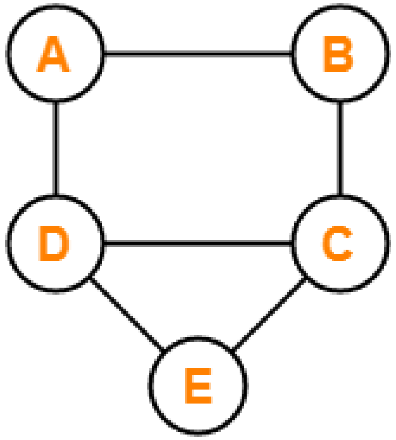
- G_2 has four vertices all of even degree, so it has a Euler circuit. The circuit is a-d-b-a-c-d-a .

❑ Hamiltonian Graphs -

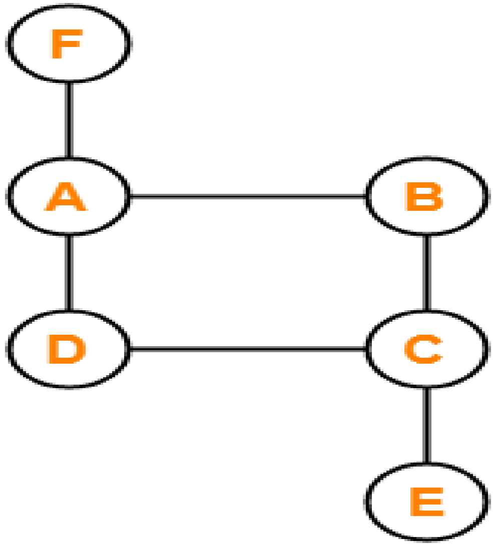
- Hamiltonian Circuit: A *Hamiltonian circuit* in a graph is a closed path that visits every vertex in the graph exactly once (such a closed loop must be a cycle).

- A Hamiltonian circuit ends up at the vertex from where it started.

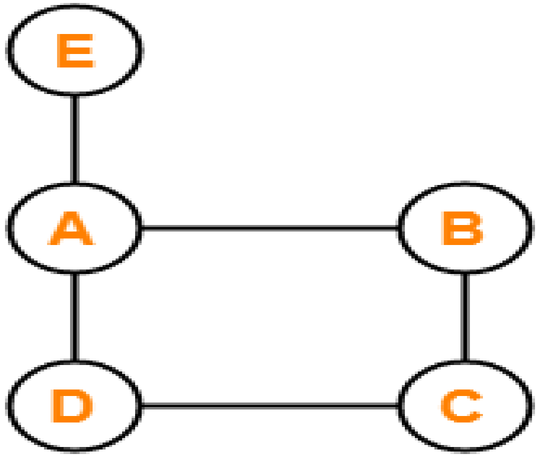
- Hamiltonian Graph: If a graph has a Hamiltonian circuit, then the graph is called a *Hamiltonian graph*.



Hamiltonian Circuit = ABCEDA



Hamiltonian Circuit Does Not Exist



amiltonian Circuit Does Not Exist

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➢ Important : An Eulerian circuit traverses every edge in a graph exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex in a graph exactly once but may repeat edges.

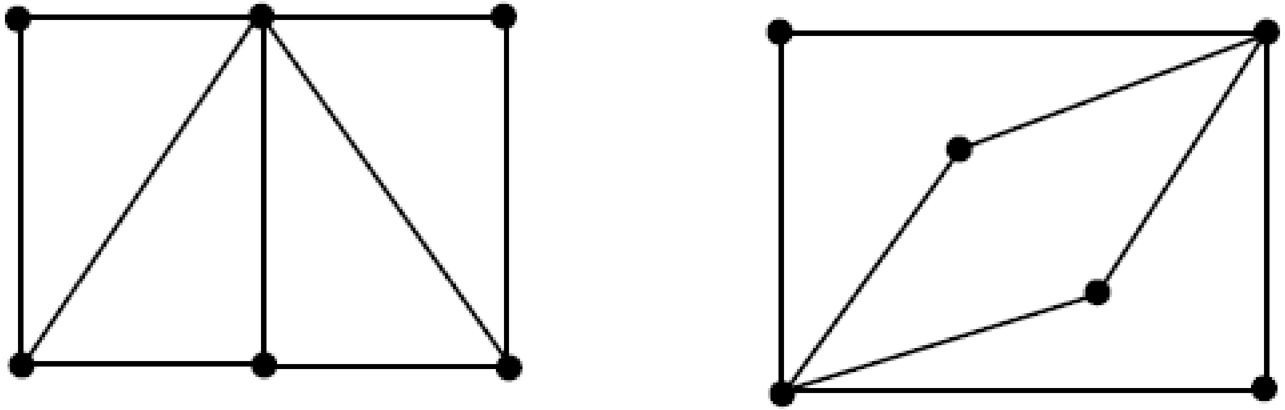
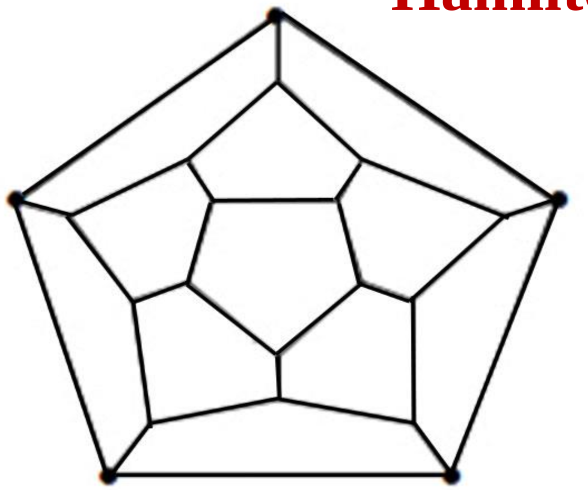


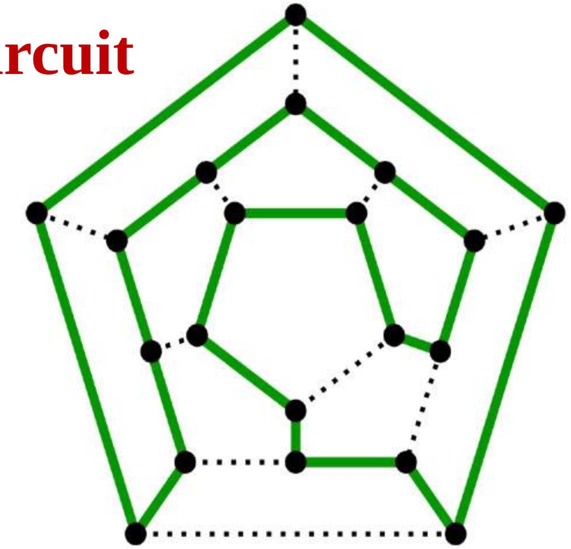
Figure 3: On the left a graph which is Hamiltonian and non-Eulerian and on the right a graph which is Eulerian and non-Hamiltonian.

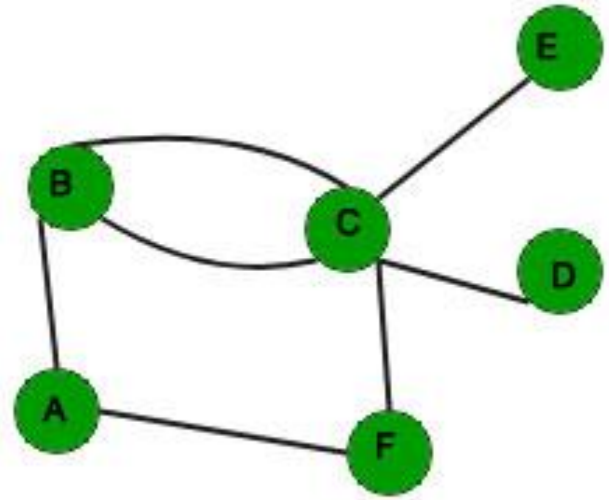
- **Dirac's Theorem-** If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamiltonian circuit.

- **Ore's Theorem-** If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G, then G has a Hamiltonian circuit.

Hamiltonian circuit







No, the graph does not have a Hamiltonian circuit as there are two vertices with degree one in the graph.

- ***All the earlier theorems for Hamiltonian graph are sufficient but not necessary conditions for the existence of a Hamiltonian circuit in a graph, there are certain graphs which have a Hamiltonian circuit but do not follow the conditions in the above-mentioned theorem.***

- For example - the cycle graph C_5 has a Hamiltonian circuit but does not follow the theorems.

- **Note -** K_n is Hamiltonian circuit for $n \geq 3$.

Thank You