Tree : Basic Concept of Tree and Its Properties

Prepared by – Subhra Sarkar
Assistant Professor, BSH Dept.
Bengal Institute of Technology

Module V – Course Contents

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- Digraph
- Matrix representation: Incidence and Adjacency matrix

- Tree: Basic concept of tree
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- Kruskal and Prim's algorithm for finding the MST
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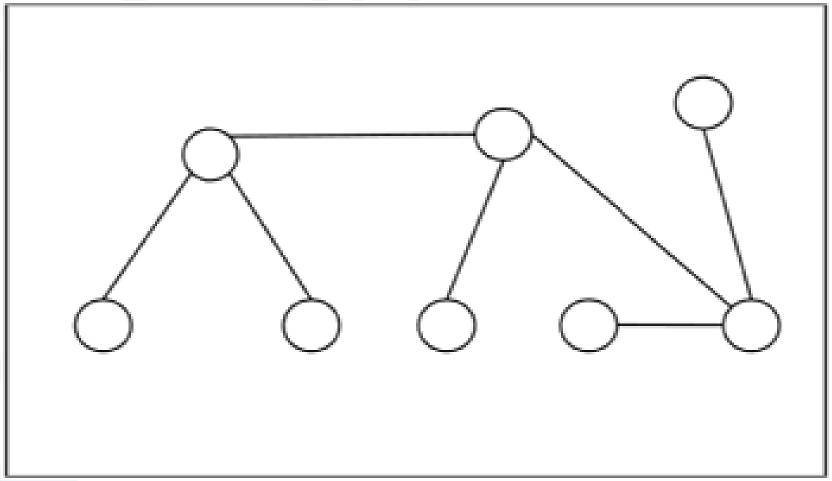
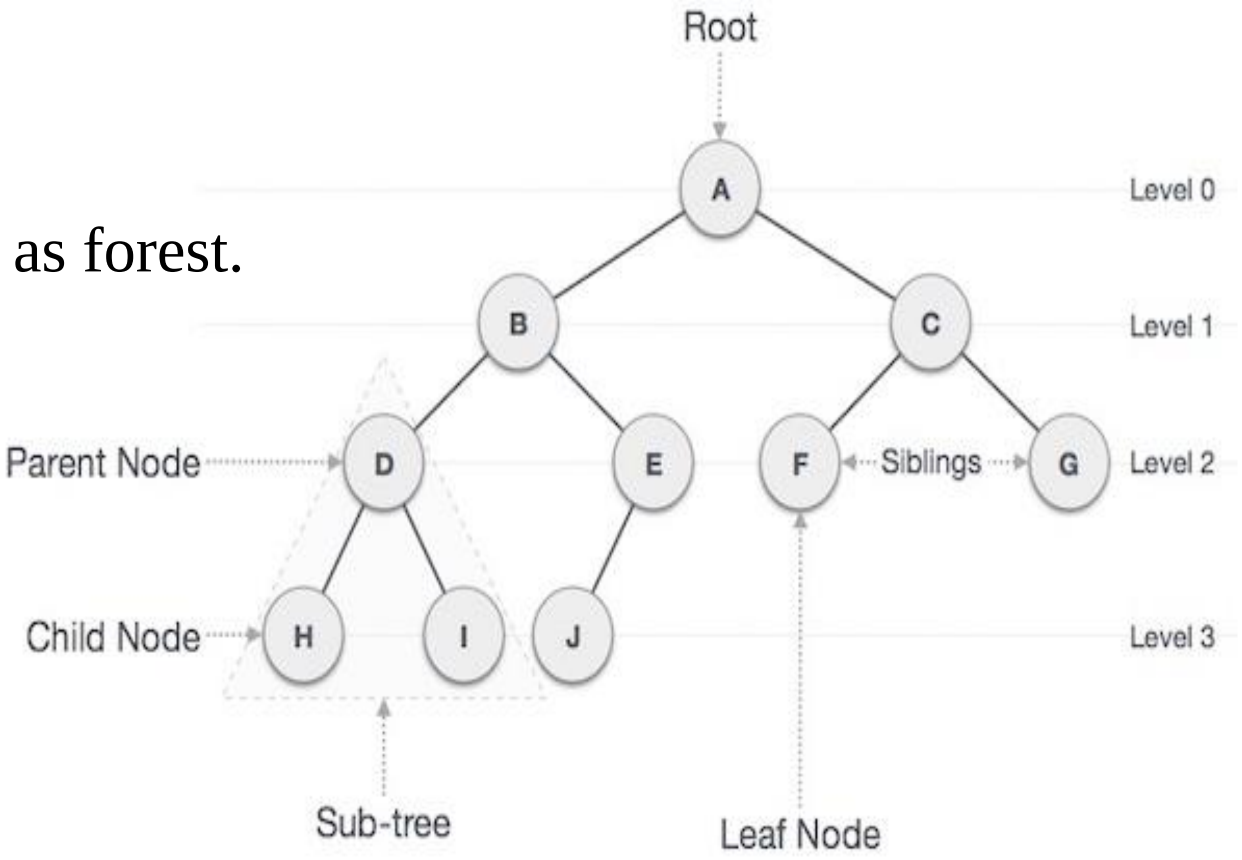
Tree
Connected & Acyclic Graph

- A connected acyclic (circuit free) graph is known as Tree.

  ✓ No loop
  ✓ No parallel edges

- Forest – A collection of some trees is known as forest.

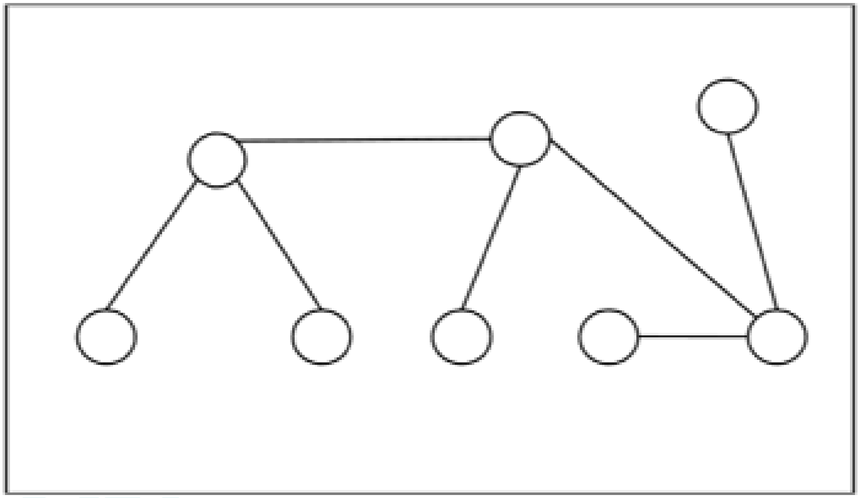




Equivalent statements for tree –

A graph G with n vertices is a Tree, then it has (n-1) edges

a. There is one and only one path between every pair of vertices of G.
b. G is connected and has (n-1) edges
c. G is acyclic and has (n-1) edges
d. G is minimally connected.
e. Addition of an edge between any two vertices in the graph G creates exactly one cycle.



- **Minimally Connected Graph –** A connected graph G is said to be minimally connected if removal of any one edge without removing the end vertices from G disconnects G.

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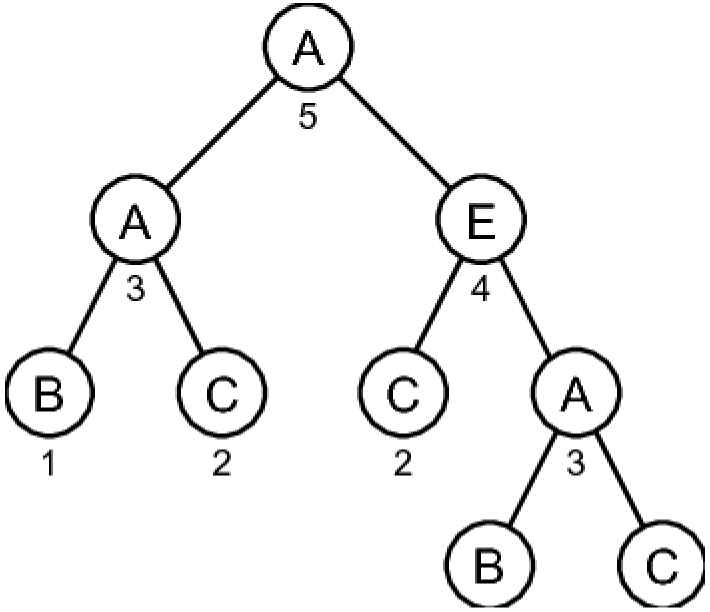
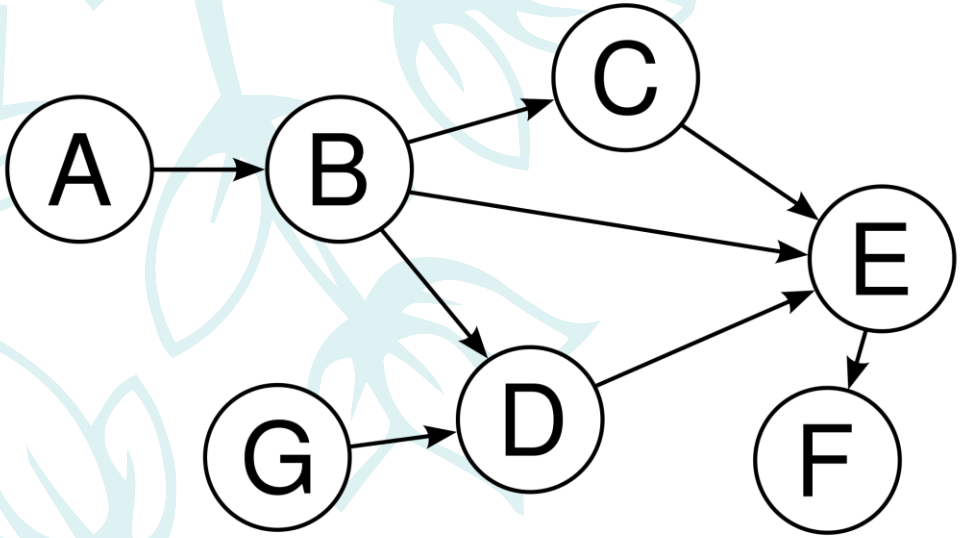
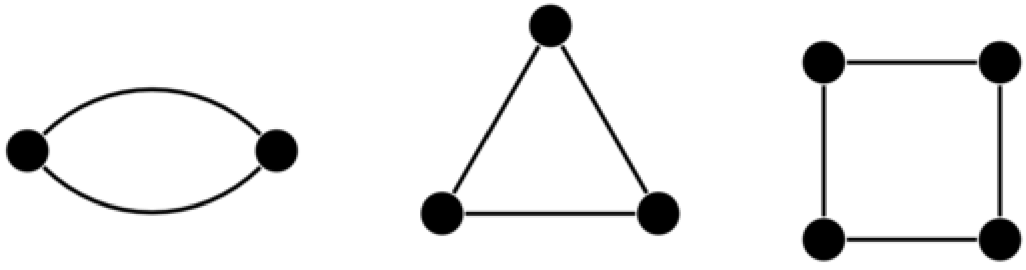
Result – There is one and only one path between every pair of vertices in a tree.

- Since T is a connected graph, there must exists at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of T, there are two distinct paths. The union of these two paths will contain a circuit and T can not be a tree.



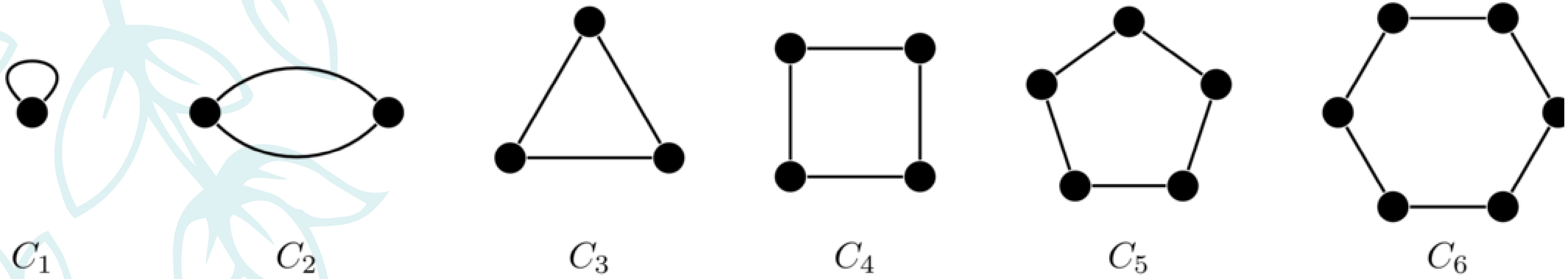




C_2 C_3 C_4

Conversely, if in a graph G there is one and only one path between every pair of vertices, then G is a tree.

- Existence of a path between every pair of vertices assures that G is connected. A circuit in a graph with two or more vertices implies that there is at least one pair of vertices a, b such that there are two distinct paths between a and b. Since G has one and only one path between every pair of vertices, G can have any circuit. Hence G is a tree.



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Result – If T is a tree with n vertices then it has precisely (n-1)edges.

- **Mathematical Induction –**

- When n = 1, i.e. T has only one vertex, then since T has no loop, T can not have any edges, i.e. it has n – 1 = 0 edges. This establish the result is true for n = 1.
- Now suppose that the result is true for n = k where k is some positive integer.
- Let T be a tree with (k+1) vertices and let u be a vertex of **degree 1** in T.
- Let e = (u, v) denote the unique edge of T which has u as an end, then if x and y are vertices in T both different from u, any path P joining x to y does not go through the vertex u, since if it did, it would involve the edge twice.
- Thus the sub graph T - u, obtained from T by deleting the vertex u (and the edge e) is connected.
- Moreover if C is a circuit in T-u then C would be a circuit in T, which is impossible, since T is a tree.
- Thus the subgraph T-u is also circuit free. Hence T-u is a tree. However T-u has k vertices (since T has k+1) and so by our induction assumption T-u has (k-1) edges.
- Since t-u has exactly one edge less than T, it follows that T has k edges as required.
- Hence assuming the result is true for k, we have shown that it is true for (k+1).
- Thus it is true for all positive integer k.

Result – Any connected graph with n vertices and (n-1) edges is a Tree.

- Consider a connected graph. We have only to show that the graph has no circuits.
- If possible, let there be a circuit in the graph with n_1 vertices so that there are n_1 edges in the circuit.
- Since the graph is connected the remaining $n - n_1$ vertices must be connected to the vertices in the circuit.
- This will require at least $n - n_1$ edges.
- Hence the total number of edges in the graph will be at least $(n - n_1) + n_1 = n$, which is impossible.
- Hence the connected graph must be circuit free. Therefore it is a tree.

Result – A circuit free graph with n vertices and (n-1) edges is a tree.

- If possible, let the graph G be disconnected having K components where $k \geq 2$ and G_1 , G_2 , …. , G_k be those components.
- Since G is acyclic, each component is acyclic and also connected. Therefore each component is a Tree.
- Let (n_i , e_i) be the number of vertices and edges respectively in the component G_i , i = 1, 2 , …, k
- Since each component is a tree $e_i = n_i - 1$.
- Therefore $\sum_{i=1}^{k} e_i = \sum_{i=1}^{k}(n_i - 1), i.e. e = n - k . But\ k \geq 2, so\ e \leq n - 2$, but total number of edges is given by e = n - 1 .
- Hence our assumption is wrong. So k = 1, the graph is a connected graph.

Result – A tree with two or more vertices has at least two pendant vertices.

- We know that, the sum of degree of vertices in any graph is 2e.
- In case of tree e = v -1, therefore the sum of degrees of vertices is 2(v-1) = 2v -2
- Since tree is connected graph that does not contain a cycle, a tree with more than one vertex cannot have any isolated vertex.
- So there must be at least two vertices of degree one in a tree, i.e. in any tree there are at least two pendant vertices.

Result - A graph is a tree if and only if it is minimally connected.

- **Necessary Condition:**

Suppose a graph G is a tree with n vertices, therefore it has (n-1) edges. If one edge is removed from G, then it has (n-2) edges and G becomes disconnected. Hence, G is a minimally connected graph.

- **Sufficient condition:**

Suppose a graph G is a minimally connected graph with n vertices, therefore the number of edges of G is $\geq$ n-1, as G is connected.
If possible, let G is not a tree then G has a circuit and it is also connected if one edge of this circuit is deleted from G. Hence, a contradiction. Therefore G is a tree.

Thank You