Problems on Multivariate Calculus

(Chain Rule and Jacobian)

1. The rectangular co-ordinates (x, y, z) of a point are given by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. Prove that the Jacobian of the transformation is $r^2\sin\theta$.

2. S.T. for the fu^n $u = x+y-z$, $v = x-y+z$, $\omega = x^2+y^2+z^2-2yz$, $J\left(\frac{u, v, \omega}{x, y, z}\right) = 0$

3. If $u = \frac{x^2+y^2+z^2}{x}$, $v = \frac{x^2+y^2+z^2}{y}$, $\omega = \frac{x^2+y^2+z^2}{z}$, find $\frac{\partial(x, y, z)}{\partial(u, v, \omega)}$. [Ans: $\frac{x^2y^2z^2}{(x^2+y^2+z^2)^3}$]

4. Investigate the extreme values of $f(x, y) = 2(x-y)^2 - x^4 - y^4$

[Ans: Extreme values $(0, 0)$
$(\sqrt{2}, -\sqrt{2})$
$(-\sqrt{2}, \sqrt{2})$]

5. Examine for maximum and minimum values of

$$f(x, y) = 2xy - 4x - 2y + x^2 + y^2 + z^2$$

6. If $u = x+y$, $v = xy$, find $\frac{\partial(u, v)}{\partial(x, y)}$ • [illegible]

7. If $u = x^2 - 2y$, $v = x+y$, prove that $\frac{\partial(u, v)}{\partial(x, y)} = 2x+2$

8. Calculate $\frac{\partial(u, v)}{\partial(x, y)}$ if $x = e^u\cos v$, $y = e^u\sin v$

9. If $u = x+y$, $v = \frac{y}{x+y}$, S.T. $\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{x+y}$

10. If $z = f(u, v) = f(x^2+2yz, y^2+2xz)$, prove that

$$(y^2 - zx)\frac{\partial v}{\partial x} + (x^2 - yz)\frac{\partial v}{\partial y} + (z^2 - xy)\frac{\partial v}{\partial z} = 0$$

11. If $\omega = \phi(2x - 3y, 3y - 4z, 4z - 2x)$, prove that
$$\frac{1}{2}\frac{\partial \omega}{\partial x} + \frac{1}{3}\frac{\partial \omega}{\partial y} + \frac{1}{4}\frac{\partial \omega}{\partial z} = 0$$

12. If $u = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$. prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

13. If $z = f(x, y)$. $x = e^{u}\cos t$. $y = e^{u}\sin t$ prove that
$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial t^2} = e^{2u}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

14. If $u = \log_e r$ and $r^2 = x^2 + y^2 + z^2$, prove that
$$r^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$