Problem Sheet - 7
(Multivariate Calculus)

1. Show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is not continuous at $(0,0)$.

2. Verify $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x-y}$ does not exist.

3. Find the following partial derivatives using defⁿ—

   a) f_x, f_y at $(0, \pi/2)$ when $f(x,y) = y\sin x$

   b) f_x, f_y ~~at~~ when $f(x,y) = e^{xy}$

4. Evaluate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when

   a) $f(x,y) = \tan^{-1}\dfrac{x^2+y^2}{x+y}$ b) $f(x,y) = x^2\tan^{-1}\dfrac{y}{x} - y^2\tan^{-1}\dfrac{x}{y}$

5. Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ Evaluate $f_x(0,0)$ and $f_y(0,0)$.

6. If $V = x^2\tan^{-1}\dfrac{y}{x}$, find $\left[\dfrac{\partial^2 V}{\partial x\,\partial y}\right]$ at $(1,1)$.

7. If $f(x,y) = (x-y)\sin(3x+2y)$, find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$ at $(0, \pi/3)$

8. If $u = \dfrac{x^2+y^2}{x+y}$, prove that $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = \left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)4$

9. Find the extrema, if any, and saddle points of the following function —

a) $x^3 + y^3 - 3axy$

b) $x^3 + y^3 - 63(x+y) + 12xy$

c) $y^2 - x^3$

d) $x^4 + y^4 - 6(x^2 + y^2) + 8xy$

10. Find the maxima and minima of $f(x, y) = x^2 + x + 2y^2$ on unit circle.

11. Determine the maximum and minimum values of $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2xz + 2yz + xy = 12$

12. Find the directional derivative of $f(x, y) = x^2 - y^2$ at $(1, 2)$ in the direction of $\frac{1}{5}(3, 4)$.

13. S.T. $(3x + 2y + 4z)\hat{i} + (2x + 5y + 4z)\hat{j} + (4x + 4y - 8z)\hat{k}$ is both solenoidal and irrotational.

14. Find grad ϕ where $\phi = 3x^2 y - y^3 z^2$ at $(1, -2, -1)$

15. Find ϕ such that $\nabla\phi = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

16. If $\vec{A} = x^2 y\,\hat{i} + y^2 z\,\hat{j} + z^2 x\,\hat{k}$, find div $\vec{A}$ at $(0, 1, 1)$

17. If $\vec{A} = 3xyz^2\,\hat{i} + 2xy^3\,\hat{j} - x^2 yz\,\hat{k}$ and $\psi = 3x^2 - yz$. Then find

i) $\vec{\nabla} \cdot \vec{A}$

ii) $\vec{A} \cdot \vec{\nabla}\psi$

iii) $\vec{\nabla} \cdot \vec{\nabla}\psi$ at $(1, -1, 1)$

18. Find div grad ϕ, where $\phi = 2x^2 y^3 z^4$

19. If $\vec{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$, show that $\vec{F} \cdot (\vec{\nabla} \times \vec{F}) = 0$

20. S.T. $\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find a scalar ϕ s.t. $\vec{F} = \vec{\nabla}\phi$