Multivariate Calculus (BS-M102)

Module V —

- ✓ Limit, continuity and partial derivatives
- ✓ Directional derivatives, Total derivative
- ✓ Tangent Plane and normal line
- ✓ Maxima, minima and saddle points
- ✓ Lagrange multipliers method
- ✓ Gradient, Curl and Divergence

Functions in Higher Dimensional Space —

We are acquainted with the meaning of a fu^n of a single real variable. Now we shall extend this idea to define a fu^n of several real variables, a fu^n of a point in Euclidean Vector space R^n. However our main concern will be with two and three - dimensional spaces.

(Simultaneous limit ~~of~~ or Double Limit)

Limits of Fu^n — A fu^n f of two independent variables x and y, is said to tend to a limit A as the point (x, y) tends to a given point (a, b), if, to each $\epsilon > 0$, $\exists\ \delta > 0$ such that $|f(x, y) - A| < \epsilon$ $\forall$ points (x, y) of the domain which belongs to some δ-nbd N of (a, b). The inequality may not be satisfied when $x = a$, $y = b$.

N may be a square nbd. $0 < |x-a| < \delta$
$0 < |y-b| < \delta$

or. N may be a circular nbd. $0 < (x-a)^2 + (y-b)^2 < \delta^2$.

or, N may be any other nbd.

Repeated or Iterated Limit –

Let $f(x, y)$ be defined in a certain nbd (a, b). Then $\lim_{x \to a} f(x, y)$, y remaining constant, when exists, will be different for different values of y and in fact this limit will be a fu^n of y, say $\phi(y)$.

If, then, $\lim_{y \to b} \phi(y)$ exists and is equal to l, we write

$$\lim_{y \to b} \left\{ \lim_{x \to a} f(x, y) \right\} = \lim_{y \to b} \phi(y) = l_1$$

l_1 is called repeated or iterated limit of $f(x, y)$ as $x \to a$ and then $y \to b$.

Thus we may obtain another repeated limit l_2 (say)

$$\lim_{x \to a} \left\{ \lim_{y \to b} f(x, y) \right\} = \lim_{x \to a} \psi(x) = l_2$$

Note – l_2 may or may not equal to l_1. Also it can happen that both repeated limit exist and have the same value, the simultaneous or double limit may not exist.

eg. Verify that the double limit $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x+y}{x-y}$ does not exist but repeated limit exist.

$\Rightarrow$ $$\lim_{(x,y) \to (0,0)} \frac{x+y}{x-y} = \lim_{(x,y) \to (0,0)} \frac{x+mx}{x-mx} \quad \left(\text{along the line } y = mx\right)$$

$$= \lim_{(x,y) \to (0,0)} \frac{1+m}{1-m}$$

$$= \frac{1+m}{1-m}$$

Thus by setting $(x, y) \to (0,0)$ along a suitable line $y = mx$, $f(x,y)$ will approach any value $\frac{1+m}{1-m}$ where m is arbitrary. Hence limit does not exist as $f(x,y)$ does not tend to a unique limit as $(x,y) \to (0,0)$.

But we can observe that both repeated limits exist.

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x+y}{x-y}\right\} = \lim_{y\to 0}\frac{y}{-y} = -1$$

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x+y}{x-y}\right\} = \lim_{x\to 0}\frac{x}{x} = 1$$

e.g. Verify that both repeated limit exist and equal to 0 for $f(x,y) = \frac{xy}{x^2+y^2}$, but double limit does not exist.

$$\Rightarrow \lim_{y\to 0}\left\{\lim_{x\to 0}\frac{xy}{x^2+y^2}\right\} = 0 = \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{xy}{x^2+y^2}\right\}$$

But $$\lim_{\substack{x\to 0\\ y\to 0}}\frac{xy}{x^2+y^2} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

along the line $y = mx$

Hence the double limit depends on the path and thus it can not take unique value as $(x,y) \to (0,0)$.

$\therefore$ Double limit does not exist.

e.g. Establish that $\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2} = 0$

$\Rightarrow$ Let $\epsilon > 0$ be given. To find a δ-nbd of $(0,0)$ $(\delta>0)$ s.t. in that nbd N, $\forall (x,y)$

$$\left| xy\,\frac{x^2-y^2}{x^2+y^2} - 0 \right| < \epsilon$$

i.e. $\left| xy\,\frac{x^2-y^2}{x^2+y^2} \right| < \epsilon$

i.e. $|x|\,|y|\,\frac{|x^2-y^2|}{x^2+y^2} < \epsilon$

now clearly $|x| < \sqrt{x^2+y^2}$ and $\frac{|x^2-y^2|}{x^2+y^2} < 1$

$|y| < \sqrt{x^2+y^2}$

Hence we can write $|x|\,|y|\,\frac{|x^2-y^2|}{x^2+y^2} < (x^2+y^2)$

$< \epsilon$

iff $0 < x^2+y^2 < \delta^2$ where $\delta = \sqrt{\epsilon}$

$\therefore$ The requirements of the definition of limit are met and hence the limit exists and equal to zero.

Note – $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Continuity of a Function of Several Variables –

A fu^n f defined in a domain $D \subseteq \mathbb{R}^n$ is said to be continuous at the pt $(x_0, y_0, \dots) \in D$ if

$$\lim_{(x,y,\dots)\to(x_0,y_0,\dots)} f(x,y,\dots) = f(x_0, y_0, \dots)$$

e.g. $f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$ The fu^n $f(x,y)$ is cont. at $(0,0)$ as $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

Partial Derivatives —

Let $u = f(x, y)$ be a fu^n of two independent variables x, y in a region R. If y is held constant then $f(x, y)$ becomes a fu^n of x alone and its derivative (if exists) is called the partial derivative of $f(x, y)$ w. r. t. x. Similarly if x is held constant, $f(x, y)$ becomes a fu^n of y alone and derivative (if exists) is called the partial derivative of $f(x, y)$ w. r. t. y. These partial derivatives, variously denoted by

$$\begin{cases} f_x(x, y) = \dfrac{\partial f}{\partial x} = u_x = \dfrac{\partial u}{\partial x} \\ f_y(x, y) = \dfrac{\partial f}{\partial y} = u_y = \dfrac{\partial u}{\partial y} \end{cases}$$

def^n —

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

If both f_x and f_y exist at $(a, b) \in R$, then we say that the fu^n $f(x, y)$ is derivable at (a, b).

Directional Derivatives in $\mathbb{R}^2$ —

Let $f(x, y)$ be a real valued fu^n of $\mathbb{R}^2$. We take $\beta = (l, m)$ a unit vector in $\mathbb{R}^2$ specifying a particular direction. A measure of rate of change in the direction β is the directional derivative.

$\therefore$ The directional derivative of f in the direction $\beta = (l, m)$ where $l^2 + m^2 = 1$ at the point (a, b) is given by $\displaystyle\lim_{t \to 0} \frac{f(a+tl, b+tm) - f(a, b)}{t}$ if it exists.

This limit is denoted by $D_\beta f(a,b)$.

e.g. Find the directional derivative of $f(x,y) = 2x^2 - xy + 5$ at $(1,1)$ in the direction of unit vector $\beta = \frac{1}{5}(3,-4)$

$$\Rightarrow D_\beta f(a,b) = D_{\left(\frac{3}{5},-\frac{4}{5}\right)} f(1,1)$$

$$= \lim_{t\to 0} \frac{f\left(1+\frac{3t}{5}, 1-\frac{4t}{5}\right) - f(1,1)}{t}$$

$$= \lim_{t\to 0} \frac{2\left(1+\frac{3t}{5}\right)^2 - \left(1+\frac{3t}{5}\right)\left(1-\frac{4t}{5}\right) + \cancel{5} - 2 + 1 - \cancel{5}}{t}$$

$$= \lim_{t\to 0} \frac{1}{t}\left[2\left(1+\frac{6t}{5}+\frac{9t^2}{5}\right) - \left(1-\frac{t}{5}-\frac{12t^2}{5}\right) - 2 + 1\right]$$

$$= \lim_{t\to 0} \frac{12}{5} + \frac{18t}{5} + \frac{1}{5} + \frac{12t}{5}$$

$$= \frac{13}{5}$$

e.g.
$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

now $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

$\therefore$ both the partial derivatives exist at $(0,0)$, but the fu^n is not continuous at $(0,0)$.

Differentiability : Total Differential –

Total differential of $f = df = f_x\, dx + f_y\, dy$

$$= \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy .$$

For a fu^n f of a single variable x, the existence of $f'(x)$ implies that f is differentiable at x and that f is continuous at x

But for a fu^n f of two independent variables x, y the existence of f_x and f_y do not imply that f is differentiable at (x, y). Also it does not always imply that f is continuous.

e.g. At $(0,0)$ the fu^n $f(x,y) = \sqrt{|xy|}$ has partial derivatives $f_x = f_y = 0$

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 = f_x(0,0)$$

$$\lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0 = f_y(0,0)$$

at $(0,0)$ the total differential $df = f_x\, dx + f_y\, dy$

$$= 0$$

now $\Delta f = f(0+h, 0+k) - f(0,0)$

$$= \sqrt{|hk|}$$

now $\dfrac{\Delta f - df}{\sqrt{h^2+k^2}} = \dfrac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} = \sqrt{\dfrac{|m|}{1+m^2}}$ does not tend to zero as it is dependent on m.

Hence $\Delta f \neq df$ $\therefore$ $f(x,y)$ is not differentiable at $(0,0)$

When $f(x,y)$ is differentiable at (x,y)

$$\Delta f = df + O\left(\sqrt{h^2+k^2}\right)$$

i.e. $$\frac{\Delta f - df}{\sqrt{h^2+k^2}} \to 0 \text{ as } (h,k)\to(0,0)$$

e.g. $$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$ is not differentiable at $(0,0)$ but is continuous at $(0,0)$ and f_x, f_y both exist at $(0,0)$.

Condition for Differentiability — (Sufficient)

If (a,b) be a point in the domain of defn of a fun f of two variables x and y s.t. one of the partial derivatives f_x or f_y exist and the other is continuous at (a,b). Suppose (i) $\frac{\partial f}{\partial x}$ exists at (a,b) and (ii) $\frac{\partial f}{\partial y}$ is continuous at (a,b) then $f(x,y)$ is differentiable at (a,b).

e.g. Let $$f(x,y) = \begin{cases} x^2\sin\frac{1}{x} + y^2\sin\frac{1}{y}, & \text{neither } x=0 \text{ nor } y=0 \\ x^2\sin\frac{1}{x}, & y=0,\ x\neq 0 \\ y^2\sin\frac{1}{y}, & x=0,\ y\neq 0 \\ 0, & x=0,\ y=0 \end{cases}$$

now $f_x(x,y) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$, $x \neq 0$

$f_x(0,y) = 0$

Similarly, $f_y(x,y) = 2y\sin\frac{1}{y} - \cos\frac{1}{y}$, $y\neq 0$

$f_y(x,0) = 0$

now $f_{xx}(0,0) = \lim\limits_{h\to 0} \dfrac{f_x(0+h,0) - f_x(0,0)}{h}$

$$= \lim_{h\to 0} \frac{f_x(h,0) - f_x(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h\sin\frac{1}{h} - \cos\frac{1}{h} - 0}{h}$$

$$= \lim_{h\to 0}\left(2\sin\frac{1}{h} - \frac{1}{h}\cos\frac{1}{h}\right)$$

= does not exist.

similarly, $f_{yy}(0,0)$ = does not exist

now $f_{xy}(0,0) = \lim\limits_{h\to 0} \dfrac{f_y(0+h,0) - f_y(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$= 0$$

eg Find $\frac{\partial u}{\partial z}$ and $\left(\frac{\partial u}{\partial z}\right)_P$ where $u(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ and $P = (1,0,-1)$

$\Rightarrow$

$$\frac{\partial u}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1} \cdot 2z$$

$$= -z(x^2+y^2+z^2)^{-3/2}$$

$$\left(\frac{\partial u}{\partial z}\right)_{P=(1,0,-1)} = -(-1)\left(1^2+0^2+(-1)^2\right)^{-3/2} = 2^{-3/2}$$

Q.8 If $u = \log(x^2+y^2)$, find $u_{xx}, u_{yy}, u_{xy}, u_{yx}$.

$\Rightarrow$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 2 - 2x \cdot 2x}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$u_y = \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$u_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\cdot 0 - 2y\cdot 2x}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

$$u_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) = -\frac{4xy}{(x^2+y^2)^2}$$

Tangent Plane and Normal Line –

Let $\phi(x,y,z) = c$ be the eq^n of a level surface. Then the eq^n of tangent plane at $P(x,y,z)$ is

$$(X-x)\frac{\partial \phi}{\partial x} + (Y-y)\frac{\partial \phi}{\partial y} + (Z-z)\frac{\partial \phi}{\partial z} = 0$$

and the eq^n of the normal at $P(x,y,z)$ is

$$\frac{X-x}{\frac{\partial \phi}{\partial x}} = \frac{Y-y}{\frac{\partial \phi}{\partial y}} = \frac{Z-z}{\frac{\partial \phi}{\partial z}}$$

e.g. Find the eq^n of the tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at the point $(2, 1, -3)$.

$\Rightarrow$ Let $\phi(x, y, z) = 2x^2 + y^2 + 2z = 3$

$$\phi_x = 4x, \quad \phi_y = 2y, \quad \phi_z = 2$$

at $(2, 1, -3)$, $\phi_x = 8$, $\phi_y = 2$, $\phi_z = 2$

$\therefore$ eq^n of tangent plane to the surface at the pt. $(2, 1, -3)$ is given by $(X - 2)\phi_x + (Y - 1)\phi_y + (Z + 3)\phi_z = 0$

$$\Rightarrow 8(x - 2) + 2(y - 1) + 2(z + 3) = 0$$

$$\Rightarrow 8x + 2y + 2z - 12 = 0$$

$$\Rightarrow 4x + y + z = 6$$

The eq^n of normal line to the surface at $(2, 1, -3)$ is

$$\frac{X - 2}{\phi_x} = \frac{Y - 1}{\phi_y} = \frac{Z + 3}{\phi_z}$$

$$\Rightarrow \frac{X - 2}{8} = \frac{Y - 1}{2} = \frac{Z + 3}{2}$$

$$\Rightarrow \frac{x - 2}{4} = y - 1 = z + 3$$