Maxima and Minima for Functions of Several Variables —

Let $z = f(x, y)$ be a fu^n of two variables. A point (x_0, y_0) is called a maximum point / minimum point of $f(x, y)$ if $\exists$ a region surrounding the point (x_0, y_0) such that $f(x, y) < f(x_0, y_0)$ / $f(x, y) > f(x_0, y_0)$ $\forall$ points (x, y) in the region.

Conditions for Extrema —

If a fu^n $z = f(x, y)$ be a continuous fu^n and possesses second order partial derivatives, ~~then~~ and $f(x, y)$ has a maxima or minima point at (x_0, y_0), then

$$\left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)} = 0 \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)} = 0$$

Now if (a, b) is a point satisfying the eq^n $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ and D is defined as

$$D = \left[f_{xx} f_{yy} - f_{xy}^2\right]_{(a,b)}$$

then I. (a, b) is maximum pt. if $D > 0$ and $f_{xx}(a, b) < 0$

II. (a, b) is minimum pt. if $D > 0$ and $f_{xx}(a, b) > 0$

III. (a, b) is neither a maximum point nor a minimum point (i.e. saddle point) if $D < 0$

IV. Further investigation is req^d if $D = 0$ (No conclusion)

Saddle points -

A point (a,b) is said to be saddle point of a fu^n $z = f(x,y)$ if $f(x,y)$ has neither maximum nor minimum at (a,b) though $\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = 0$

and $\frac{\partial f}{\partial y}(a,b) = f_y(a,b) = 0$

eg Determine the critical points and locate any maxima minima and saddle points of $f(x,y) = 2x^2 - 4xy + y^4 + 2$

$\Rightarrow$ $$f(x,y) = 2x^2 - 4xy + y^4 + 2$$

Step I : To find critical points by solving $f_x = 0 = f_y$

$$\frac{\partial f}{\partial x} = 4x - 4y$$

$$\frac{\partial f}{\partial y} = -4x + 4y^3$$

now $f_x = 0$ $\Rightarrow x = y$ | $f_y = 0$ $\Rightarrow x = y^3$ $\Rightarrow y = y^3$ $\Rightarrow y = 0, \pm 1$

Therefore critical points are $(0,0)$, $(1,1)$, $(-1,-1)$

Step II : Checking for Extrema -

$A = f_{xx} = 4$ $\qquad C = f_{xy} = -4$

$B = f_{yy} = 12y^2$

$$\therefore D = f_{xx} f_{yy} - f_{xy}^2 = AB - C^2$$
$$= 48y^2 - 16 = 16(3y^2 - 1)$$

$D(0,0) = -16 < 0$

$A = f_{xx}(0,0) = 4 > 0$

$\therefore (0,0)$ is a saddle point for $f(x,y)$

$D(1,1) = 32 > 0$

$A = f_{xx}(1,1) = 4 > 0$

$\therefore (1,1)$ is a minima of $f(x,y)$

$D(-1,-1) = 32 > 0$

$A(-1,-1) = 4 > 0$

$\therefore (-1,-1)$ is a minima of $f(x,y)$

Q.2 Determine the critical points of $f(x,y) = x^3 - 12x + y^3 + 3y^2 - 9y$

$\rightarrow$

$$f = x^3 - 12x + y^3 + 3y^2 - 9y$$

$$f_x = 3x^2 - 12$$

$$f_y = 3y^2 + 6y - 9$$

$$f_{xx} = 6x$$

$$f_{yy} = 6y + 6$$

$$f_{xy} = 0$$

Step I :- Set $f_x = 0 = f_y$. then we get

$x^2 = 4 \Rightarrow x = \pm 2$

$y^2 + 2y - 3 = 0$

$\Rightarrow (y+3)(y-1) = 0$

$\Rightarrow y = -3, 1$

$\therefore$ critical points are $(2,-3)$, $(2,1)$, $(-2,-3)$, $(-2,1)$

Step II - $D = f_{xx} f_{xy} - f_{xy}^2 = 36x(y+1)$

$D(2,-3) < 0$

$f_{xx}(2,-3) > 0$

$\therefore f$ has maxima at $(2,-3)$

$D(2,1) > 0$

$f_{xx}(2,1) > 0$

$\therefore f$ has minima at $(2,1)$

$D(-2,-3) > 0$

$f_{xx}(-2,-3) < 0$

$\therefore$ No conclusion

$D(-2,1) < 0$: f has saddle point at $(-2,1)$

Lagrange Multiplier Method –

We want to optimize i.e. finding minimum and maximum value of a fu^n $f(x,y,z)$ subject to the constraint $g(x,y,z) = k$.

Step I – Solve the following system of eq^n

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$g(x,y,z) = k$$

Step II – Plug in all solutions (x,y,z) from the first step into $f(x,y,z)$ and identify the minimum and maximum values, provided they exist and $\nabla g \neq \vec{0}$ at the point.

The constant, λ, is called the Lagrange Multiplier.

Q. Find the maximum and minimum values of $f(x,y) = 81x^2 + y^2$ subject to the constraint $4x^2 + y^2 = 9$.

→ Consider a fu^n. $h(x,y) = f(x,y) - \lambda g(x,y)$

$$= 81x^2 + y^2 - \lambda(4x^2 + y^2 - 9)$$

now $h_x = 162x - 8\lambda x$

$h_y = 2y - 2\lambda y$

Step I – Set $h_x = 0 = h_y$

$\therefore$ $162x - 8\lambda x = 0$

$\Rightarrow x(162 - 8\lambda) = 0$

$\Rightarrow x = 0$ or $\lambda = \frac{162}{8}$

if $x = 0$, then $4x^2 + y^2 = 9$

$\Rightarrow y = \pm 3$

points are $(0, 3), (0, -3)$

$2y - 2\lambda y = 0$

$\Rightarrow y = 0$ or $\lambda = 1$

if $y = 0$, then $4x^2 + y^2 = 9$

$\Rightarrow x^2 = 9/4$

$\Rightarrow x = \pm \frac{3}{2}$

points are —

$\left(\frac{3}{2}, 0\right), \left(-\frac{3}{2}, 0\right)$

Step II –

now $f(0, 3) = 9$

$f(0, -3) = 9$

$\therefore$ Minima of $f(x, y)$ occurs at $(0, 3)$ and $(0, -3)$.

$f\left(\frac{3}{2}, 0\right) = \frac{729}{4} > 9$

$f\left(-\frac{3}{2}, 0\right) = \frac{729}{4}$

$\therefore$ Maxima of $f(x, y)$ occurs at $(x, y) = \left(\frac{3}{2}, 0\right)$ and $(x, y) = \left(-\frac{3}{2}, 0\right)$

Q. Find the minima and maxima of $f(x, y, z) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$

$\Rightarrow$ Step I – $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

or. $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 0 = \lambda \cdot 2x$

$\Rightarrow$ either $x = 0$ or $\lambda \neq 0$ — (I)

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$

$\Rightarrow 2y = \lambda \cdot 2y \Rightarrow \lambda = 1$ or $y = 0$ — (II) and $\dfrac{\partial f}{\partial z} = \lambda \dfrac{\partial g}{\partial z}$

$\Rightarrow -10 = \lambda \cdot 2z$

$\Rightarrow \lambda z = -5$ —— (iii)

now $g(x, y, z) = 36 = x^2 + y^2 + z^2$

from eq^n (ii), $y = 0$ $\quad \therefore z^2 = 36 \Rightarrow z = \pm 6$

from eq^n (i), $x = 0$

$\therefore$ points are $(0, 0, 6), (0, 0, -6)$

now from eq^n (iii), $\lambda z = -5$

from eq^n (ii), one possible sol^n is $\lambda = 1$

$\therefore z = -5$

Hence $x^2 + y^2 + z^2 = 36 \Rightarrow 0 + y^2 + 25 = 36$

$\Rightarrow y = \pm\sqrt{11}$

$\therefore$ points are $(0, \sqrt{11}, -5), (0, -\sqrt{11}, -5)$

Step II –

now $f(x, y, z) = y^2 - 10z$

at $(0, 0, 6)$, $f = \underline{-60} < 0$

at $(0, 0, -6)$, $f = 60 > 0$

at $(0, \sqrt{11}, -5)$, $f = 11 + 50 = \underline{61} > 0$

at $(0, -\sqrt{11}, -5)$, $f = 11 + 50 = 61 > 0$

$\therefore$ maxima occurs at $(0, \sqrt{11}, -5)$ and $(0, -\sqrt{11}, -5)$ and maximum value is 61. Also minima occurs at $(0, 0, 6)$ and minimum values is -60.

Q: Determine the maxima and minima of $f(x, y, z) = 2x + 3y + z$ subject to the constraint $x^2 + y^2 + z^2 - 1 = 0$

$\Rightarrow$ Let $f(x, y, z) = 2x + 3y + z$

$g(x, y, z) = x^2 + y^2 + z^2 = 1$

Step I – $\nabla f = \lambda \nabla g$, then

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$

$\Rightarrow 2 = 2x\lambda$

$\Rightarrow x\lambda = 1$

$\Rightarrow x = 1/\lambda$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$

$\Rightarrow 3 = 2y\lambda$

$\Rightarrow y\lambda = \frac{3}{2}$

$\Rightarrow y = \frac{3}{2\lambda}$

$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}$

$\Rightarrow 1 = 2z\lambda$

$\Rightarrow z\lambda = \frac{1}{2}$

$\Rightarrow z = 1/2\lambda$

now $g(x, y, z) = 1 \Rightarrow x^2 + y^2 + z^2 = 1$

$$\Rightarrow \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

$$\Rightarrow \frac{14}{4\lambda^2} = 1 \Rightarrow \lambda^2 = \frac{7}{2}$$

$$\Rightarrow \lambda = \pm\sqrt{7/2}$$

Therefore points are (x, y, z)

$$= \left(\frac{1}{\lambda}, \frac{3}{2\lambda}, \frac{1}{2\lambda}\right)$$

$$= \left(\sqrt{\frac{2}{7}}, \frac{3}{2}\sqrt{\frac{2}{7}}, \frac{1}{2}\sqrt{\frac{3}{7}}\right) \text{ and}$$

$$\left(-\sqrt{\frac{2}{7}}, -\frac{3}{2}\sqrt{\frac{2}{7}}, -\frac{1}{2}\sqrt{\frac{2}{7}}\right)$$

now at $\left(\sqrt{\frac{2}{7}}, \frac{3}{2}\sqrt{\frac{2}{7}}, \frac{1}{2}\sqrt{\frac{2}{7}}\right)$, value of f

$$= 2\sqrt{\frac{2}{7}} + \frac{9}{2}\sqrt{\frac{2}{7}} + \frac{1}{2}\sqrt{\frac{2}{7}}$$

$$= 7\sqrt{\frac{2}{7}} \quad (\text{maxima})$$

at $\left(-\sqrt{\frac{2}{7}}, -\frac{3}{2}\sqrt{\frac{2}{7}}, -\frac{1}{2}\sqrt{\frac{2}{7}}\right)$. value of f

$$= -2\sqrt{\frac{2}{7}} - \frac{9}{2}\sqrt{\frac{2}{7}} - \frac{1}{2}\sqrt{\frac{2}{7}}$$

$$= -7\sqrt{\frac{2}{7}} \quad (\text{minima})$$