Countable and Uncountable Sets

A set A is said to be *finite*, if A is empty or there is $n \in \mathbb{N}$ and there is a bijection $f : \{1, \dots, n\} \to A$. Otherwise the set A is called *infinite*. Two sets A and B are called *equinumerous*, written $A \sim B$, if there is a bijection $f : X \to Y$. A set A is called *countably infinite* if $A \sim \mathbb{N}$. We say that A is countable if $A \sim \mathbb{N}$ or A is finite.

Example 3.1. The sets $(0, \infty)$ and $\mathbb{R}$ are equinumerous. Indeed, the function $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$ is a bijection.

Example 3.2. The set $\mathbb{Z}$ of integers is countably infinite. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Then f is a bijection from $\mathbb{N}$ to $\mathbb{Z}$ so that $\mathbb{N} \sim \mathbb{Z}$.

If there is no bijection between $\mathbb{N}$ and A, then A is called *uncountable*.

Theorem 3.3. *There is no surjection from a set A to $\mathcal{P}(A)$.*

Proof. Consider any function $f : A \to \mathcal{P}(A)$ and let

$$B = \{a \in A |\, a \notin f(a)\}.$$

We claim that there is no $b \in A$ such that $f(b) = B$. Indeed, assume $f(b) = B$ for some $b \in A$. Then either $b \in B$ hence $b \notin f(b)$ which is a contradiction, or $b \notin B = f(b)$ implying that $b \in B$ which is again a contradiction. Hence the map f is not surjective as claimed. ■

As a corollary we have the following result.

Corollary 3.4. *The set $\mathcal{P}(\mathbb{N})$ is uncountable.*

Proposition 3.5. *Any subset of a countable set is countable.*

Proof. Without loss of generality we may assume that A is an infinite subset of $\mathbb{N}$. We define $h : \mathbb{N} \to A$ as follows. Let $h(1) = \min A$. Since A is infinite, A is nonempty and so $h()$ is well-defined. Having defined $h(n-1)$, we define $h(n) = \min(A \setminus \{h(1), \dots, h(n-1)\})$. Again since A is infinite the set $(A \setminus \{h(1), \dots, h(n-1)\})$ is nonempty, $h(n)$ is well-defined. We claim that h is a bijection. We first show that h is an injection. To see this we prove that $h(n+k) > h(n)$ for all $n, k \in \mathbb{N}$. By construction $h(n+1) > h(n)$

for all $n \in \mathbb{N}$. Then setting $B = \{k \in \mathbb{N}|\, h(n+k) > h(n)\}$ we see that $1 \in B$ and if $h(n+(k-1)) > h(n)$, then $h(n+k) > h(n+(k-1)) > h(n)$. Consequently, $B = \mathbb{N}$. Since n was arbitrary, $h(n+k) > h(n)$ for all $n, k \in \mathbb{N}$. Now taking distinct $n, m \in \mathbb{N}$ we may assume that $m > n$ so that $m = n + k$. By the above $h(m) = h(n+k) > h(n)$ proving that h is an injection. Next we show that h is a surjection. To do this we first show that $h(n) \geq n$. Let $C = \{n \in \mathbb{N}|\, h(n) \geq n\}$. Clearly, $1 \in C$. If $k \in C$, then $h(k+1) > h(k) \geq n$ so that $h(k+1) \geq k+1$. Hence $k+1 \in C$ and by the principle of mathematical induction $C = \mathbb{N}$. Now take $n_0 \in A$. We have to show that $h(m_0) = n_0$ for some $m_0 \in \mathbb{N}$. If $n_0 = 1$, then $m_0 = 1$. So assume that $n_0 \geq 2$. Consider the set $D = \{n \in A|\, h(n) \geq n_0\}$. Since $h(n_0) \geq n_0$, the set D is nonempty and by the well-ordering principle D has a minimum. Let $m_0 = \min D$. If $m_0 = 1$, then $h(m_0) = \min A \leq n_0 \leq h(m_0)$ and hence $h(m_0) = n_0$. So we may also assume that $n_> \min A$. Then $h(m_0) \geq n_0 > h(m_0 - 1) > \ldots > h(1)$ in view of definitions of m_0 and h. Since $h(m_0) = \min(A \setminus \{h(1), \ldots, h(m_0-1)\})$ and $n_0 \in A \setminus \{h(1), \ldots, h(m_0-1)\}$ and $h(m_0) \geq n_0$, it follows that $h(m_0) = n_0$. This proves that h is also a surjection. ■

Proposition 3.6. *Let A be a non-empty set. Then the following are equivalent.*

(a) A is countable.

(b) There exists a surjection $f : \mathbb{N} \to A$.

(c) There exists an injection $g : A \to \mathbb{N}$.

Proof. $(a) \implies (b)$ If A is countably infinite, then there exists a bijection $f : \mathbb{N} \to A$ and then (b) follows. If A is finite, then there is bijection $h : \{1, \ldots, n\} \to A$ for some n. Then the function $f : \mathbb{N} \to A$ defined by

$$f(i) = \begin{cases} h(i) & 1 \leq i \leq n, \\ h(n) & i > n. \end{cases}$$

is a surjection.
$(b) \implies (c)$. Assume that $f : \mathbb{N} \to A$ is a surjection. We claim that there is an injection $g; A \to \mathbb{N}$. To define g note that if $a \in A$, then $f^{-1}(\{a\}) \neq \emptyset$. Hence we set $g(a) = \min f^{-1}((a)\})$. Now note that if $a \neq a'$, then the sets $f^{-1}(\{a\}) \cap f^{-1}(\{a'\}) = \emptyset$ which implies $\min^{-1}(\{a\}) \neq \min^{-1}(\{a'\})$. Hence $g(a) \neq g(a')$and $g : A \to \mathbb{N}$ is an injective.
$(c) \Rightarrow (a)$. Assume that $g : A \to \mathbb{N}$ is an injection. We want to show that A

is countable. Since $g : A \to g(A)$ is a bijection and $g(A) \subset \mathbb{N}$, Proposition 3.5 implies that A is countable. ■

Corollary 3.7. *The set* $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof. By Proposition 3.6 it suffices to construct an injective function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(n, m) = 2^n 3^m$. Assume that $2^n 3^m = 2^k 3^l$. If $n < k$, then $3^m = 2^{k-n} 3^l$. The left side of this equality is an odd number whereas the right is an even number implying $n = k$ and $3^m = 3^l$. Then also $m = l$. Hence f is injective. ■

Proposition 3.8. *If* A *and* B *are countable, then* $A \times B$ *is countable.*

Proof. Since A and B are countable, there exist surjective functions $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by $F(n, m) = (f(n), g(m))$. The function F is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countably infinite, there is a bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then $G : \mathbb{N} \times A \times B$ defined by $G = F \circ h$ is a surjection. By part (c) of Proposition 3.6, the set $A \times B$ $A \times B$ is countable. ■

Corollary 3.9. *The set* $\mathbb{Q}$ *of all rational numbers is countable.*

Proposition 3.10. *Assume that the set* I *is countable and* A_i *is countable for every* $i \in I$. *Then* $\bigcup_{i \in I} A_i$ *is countable.*

Proof. For each $i \in I$, there exists a surjection $f_i : \mathbb{N} \to A_i$. Moreover, since I is countable, there exists a surjection $g : \mathbb{N} \to I$. Now define $h : \mathbb{N} \times \mathbb{N} \to \bigcup_{i \in I} A_i$ by $F(n, m) = f_{g(n)}(m)$ and let $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. Then F is a surjection and the composition $G = F \circ h : \mathbb{N} \to \bigcup_{i \in I} A_i$ is a surjection. By Proposition 3.6, $\bigcup_{i \in i} A_i$ is countable. ■

Proposition 3.11. *The set of real numbers* $\mathbb{R}$ *is uncountable.*

The proof will be a consequence of the following result about nested intervals.

Proposition 3.12. *Assume that* $(I_n)_{n \in \mathbb{N}}$ *is a countable collection of closed and bounded intervals* $I_n = a_n, b_n]$ *satisfying* $I_{n+1} \subset I_n$ *for all* $n \in \mathbb{N}$. *Then* $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Proof. Since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n, it follows that $a_n \leq b_k$ for all $n, k \in \mathbb{N}$. So, the set $A = \{a_n | \, n \in \mathbb{N}\}$ is bounded above by every b_k and consequently $a := \sup A \leq b_k$ for all $k \in \mathbb{N}$. But this implies that the set $B = \{b_k | \, k \in \mathbb{N}\}$ is bounded below by a so that $a \leq b := \inf B$. Hence $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$. ■

Proof of Proposition 3.11. Arguing by contradiction assume that $\mathbb{R}$ is countable. Let $x_1, x_2, x_3, \ldots$ be enumeration of $\mathbb{R}$. Choose a closed bounded interval I_1 such that $x_1 \notin I_1$. Having chosen the closed intervals $I_1, I_2, \ldots, I_{n-1}$, we choose the closed interval I_n to be a subset of I_{n-1} such that $x_n \notin I_n$. Consequently, we have a countable collection of closed bounded intervals (I_n) such that $I_{n+1} \subset I_n$ and $x_n \notin I_n$. Then by the above proposition, $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. Observe that if x belongs to this intersection, then x is not on the list $x_1, x_2, \ldots$, contradiction.

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Lemma 1.1 *If S is both countable and infinite, then there is a bijection between S and $\boldsymbol{N}$ itself.*

Proof: For any $s \in S$, we let $f(s)$ denote the value of k such that s is the kth smallest element of S. This map is well defined for any s, because there are only finitely many natural numbers between 1 and s. It is impossible for two different elements of S to both be the kth smallest element of S. Hence f is one-to-one. Also, since S is infinite, f is onto. ♠

Lemma 1.2 *If S is countable and $S' \subset S$, then S is also countable.*

Proof: Since S is countable, there is a bijection $f : S \to \boldsymbol{N}$. But then $f(S') = \boldsymbol{N}'$ is a subset of $\boldsymbol{N}$, and f is a bijection between S' and $\boldsymbol{N}'$. ♠

A set is called *uncountable* if it is not countable. One of the things I will do below is show the existence of uncountable sets.

Lemma 1.3 *If $S' \subset S$ and S' is uncountable, then so is S.*

Proof: This is an immediate consequence of the previous result. If S is countable, then so is S'. But S' is uncountable. So, S is uncountable as well. ♠

2 Examples of Countable Sets

Finite sets are countable sets. In this section, I'll concentrate on examples of countably infinite sets.

2.1 The Integers

The integers $\boldsymbol{Z}$ form a countable set. A bijection from $\boldsymbol{Z}$ to $\boldsymbol{N}$ is given by $f(k) + 2k$ if $k \geq 0$ and $f(k) = 2(-k) + 1$ if $k < 0$. So, f maps $0, 1, 2, 3...$ to $0, 2, 4, 6...$ and f maps $-1, -2, -3, -4...$ to $1, 3, 5, 7....$

2.2 The Rational Numbers

I'll give a different argument than the one I gave in class. Let L_q denote the finite list of all rational numbers between $-q$ and q that have denominator at most q. There are at most $q(2q+1)$ elements of L_q. We can make the list $L_1, L_2, L_3, ...$ and throw out repeaters. This makes a list of all the rational numbers. As above, we define $f(p/q)$ to be the value of k such that p/q is the kth fraction on our list.

2.3 The Algebraic Numbers

A real number x is called *algebraic* if x is the root of a polynomial equation $c_0 + c_1x + ... + c_nx^n$ where all the c's are integers. For instance, $\sqrt{2}$ is an algebraic integer because it is a root of the equation $x^2 - 2 = 0$. To show that the set of algebraic numbers is countable, let L_k denote the set of algebraic numbers that satisfy polynomials of the form $c_0 + c_1x + ... + c_nx^n$ where $n < k$ and $\max(|c_j|) < k$. Note that there are at most k^k polynomials of this form, and each one has at most k roots. Hence L_k is a finite set having at most k^{k+1} elements. As above, we make our list L_1, L_2, L_3 of all algebraic numbers and weed out repeaters.

2.4 Countable Unions of Countable Sets

Lemma 2.1 *Suppose that $S_1, S_2, ... \subset T$ are disjoint countable sets. Then $S = \bigcup_i S_i$ is a countable set.*

Proof: There are bijections $f_i : S_i \to \boldsymbol{N}$ for each i. Let L_k denote the set of elements $s \in S$ such that s lies in some S_i for $i < k$, and $f_i(s) < k$. Note that L_k is a finite set. It has at most k^2 members. The list $L_1, L_2, L_3...$ contains every element of S. Weeding out repeaters, as above, we see that we have listed all the elements of S. Hence S is countable. ♠

The same result holds even if the sets S_i are not disjoint. In the general case, we would define

$$S'_k = S_k - \bigcup_{i=1}^{k-1} S_i,$$

and apply the above argument to the sets $S'_1, S'_2....$ The point is that S'_i is countable, the various S' sets are disjoint, and $\bigcup_i S_i = \bigcup_i S'_i$.

3 Examples of Uncountable Sets

3.1 The Set of Binary Sequences

Let S denote the set of infinite binary sequences. Here is Cantor's famous proof that S is an uncountable set. Suppose that $f : S \to \boldsymbol{N}$ is a bijection. We form a new binary sequence A by declaring that the nth digit of A is the opposite of the nth digit of $f^{-1}(n)$. The idea here is that $f^{-1}(n)$ is some binary sequence and we can look at its nth digit and reverse it.

Supposedly, there is some N such that $f(A) = N$. But then the Nth digit of $A = f^{-1}(N)$ is the opposite of the Nth digit of A, and this is a contradiction.

3.2 The Real Numbers

Let $\boldsymbol{R}$ denote the reals. Let $\boldsymbol{R}'$ denote the set of real numbers, between 0 and 1, having decimal expansions that only involve 3s and 7s. (This set $\boldsymbol{R}'$ is an example of what is called a *Cantor set*.) There is a bijection between $\boldsymbol{R}'$ and the set S of infinite binary sequences. For instance, the sequence 0101001... is mapped to .3737337.... Hence $\boldsymbol{R}'$ is uncountable. But then Lemma 1.3 says that $\boldsymbol{R}$ is uncountable as well.

3.3 The Transcendental Numbers

A real number x is called *transcendental* if x is not an algebraic number. Let $\boldsymbol{A}$ denote the set of algebraic numbers and let $\boldsymbol{T}$ denote the set of transcendental numbers. Note that $\boldsymbol{R} = \boldsymbol{A} \cup \boldsymbol{T}$ and $\boldsymbol{A}$ is countable. If $\boldsymbol{T}$ were countable then $\boldsymbol{R}$ would be the union of two countable sets. Since $\boldsymbol{R}$ is uncountable, $\boldsymbol{R}$ is not the union of two countable sets. Hence $\boldsymbol{T}$ is uncountable.

The upshot of this argument is that *there are many more transcendental numbers than algebraic numbers*.

3.4 Tail Ends of Binary Sequences

Let T denote the set of binary sequences. We say that two binary sequences A_1 and A_2 are *equivalent* if they have the same tail end. For instance 1001111... and 111111... are equivalent.