Chapter

2

Mathematical Logic

- Sentences
- Statements
- Propositional function Quantifiers
- Connectives
- Type of sentences
- Truth value and truth table
- Tautology
- Contradiction
- Validity of arguments
- Rule of inferences
- Logical identities

2.1 INTRODUCTION

Mathematical logic deals with the topics like statements, negation, connectives, compound statements, conjunction, disjunction, duality, truth table, conditional and bi-conditional statements, valid arguments, tautologies, etc.

In the algebra of sets it has been observed that there are some ceratin primitive concepts associated with undefined terms. The terms true, false, and proposition (statement) are taken here as undefined.

A statement is a declarative sentence which has one and only one of two possible values. These two values are 'true' and 'false' or in other words statements is a declarative sentence which is either true or false but not both. The truth values 'true' and 'false' are denoted by the symbol T and F respectively. Sometimes it is also denoted by 1 (for true) and 0 (for false). The following examples are typical propositions :

(1) 7 is a prime number

(2) When 5 is added to 6 the sum is 7

(3) Living creatures exist on the planet Venus.

In the above examples, (1) is true, (2) is false, while (3) is either true or false but not both. Thus above sentences are statements. Let us consider other examples :

(1) May God bless you ! (a wish)

(2) Please wait here. (a request)

(3) What are you doing ? (an enquiry)

(4) May you live long.

In the above example, we can observe that no truth values can be given to the sentences. That is, they do not declare a definite truth value T or F. Thus they are not statements.

If we assume that the statement is true, then from its content we have that it is false while if the statement is assumed to be false, then from its content we have that it is true.

2.1.1 Statement Letters

It is well known fact that symbols have played a key role in any field, either a field of mathematics or a field or science. Therefore, they have great importance in Mathematics.

Definition : *The symbols, which are taken to represent the statement, are called statement letters. Therefore, we shall use lower case letters p, q, r, ... to represent statement.*

For example : If the statement 'Delhi is the capital of India', is denoted by the letter 'p', then it can be written as

p = Delhi is the capital of India.

From any statement or set of statements, other statements may be formed. The simplest example is that of forming from the statement p the negation of p which is denoted by p'. For any statement p, we define p' or $\neg p$ to be the statement "it is false that p". For example, suppose that p is the statement

p = sleeping is pleasant

The negation of this statement would be the statement.

"it is false that sleeping is pleasant"

or in other words, we can write the negation of p as follows :

"sleeping is not pleasant"

or "sleeping is unpleasant"

2.1.2 Open Statement

A sentence having one or more than one variables becomes a statement after giving some certain value to the variables, is called open statement.

For example : Let us consider $3x + y = 7$.

If we put $x = 2$ and $y = 1$, then above sentence becomes a true statement. Such a sentence is called an open statement.

2.2 PROPOSITIONAL FUNCTIONS QUANTIFIERS

A proposition is a declarative sentence that is either true or false but not both.

Let A be a given set. A propositional function defined on A is an expression $p(x)$, which has the property that $p(a)$ is true or false for each $a \in A$, *i.e.*, $p(x)$ becomes a statement whenever any element $a \in A$ is substituted for the variable x. The set A is called **domain** of $p(x)$ and the set T_p of all the elements of A for which $p(a)$ is True is called the truth set of $p(x)$, *i.e.*,

$$T_p = \{x : x \in A,\ p(x) \text{ is true}\}$$

For example : Let $p(x) = x + 4 > 1$. Its true set is

$$\{x : x \in \mathbf{N} : x + 4 > 1\} = \mathbf{N}$$

Thus, $p(x)$ is true for every element in **N**.

2.2.1 Universal Quantifiers

Let $p(x)$ be a propositional function defined on a set A. Consider the expression $[\forall\ x \in A,\ p(x) \text{ is true}]$. then the symbol $\forall$ (for all) is called the universal quantifier.

2.2.2 Existential Quantifiers

Let $p(x)$ be a propositional function defined on a set A. Consider the expression $[\exists\ x \in A,\ p(x) \text{ is true}]$. then the symbol $\exists$ (there exists) is called the existential quantifier.

2.2.3 Multiple Quantifiers

If a proposition have more than one variable then we can quantify it more than once.

For example : $p(x, y) = x^2 - y^2 = (x - y)(x + y)$

Multiple universal quantifiers can be arranged in any order without logically changing the meaning of resulting propositions. The same is true for the multiple existential quantifiers.

For example :

$$p(x, y) : x + y = 4 \text{ and } x - y = 2$$
$$= [\exists x \in \mathbf{R}, \exists y \in \mathbf{R} : x + y = 4 \text{ and } x - y = 2]$$

ILLUSTRATIVE EXAMPLES

1. *If $p(x)$ is a formula in x, then translate the following :*

(a) $\forall x, [p(x)]$ *(b)* $\exists x, [p(x)]$ *(c)* $\forall x [\sim p(x)]$ *(d)* $\exists x [\sim p(x)]$

Solution. (a) Every x has the property $p(x)$.

(b) There exist x which has property $p(x)$.

(c) No x has property $p(x)$

(d) There is some x which do not have the property $p(x)$.

2. *Translate the following statements, involving quantifiers into formulae :*

(a) All rationals are real *(b) No rationals are real*

(c) Some rationals are real *(d) Some rationals are not real.*

Solution. Let x be an individual variable and $Q(x) \equiv x$ is rational $R(x) \equiv x$ is real. Then,

(a) $\forall x, [Q(x) \rightarrow R(x)]$ (b) $\forall x \sim [Q(x) \rightarrow R(x)]$

(c) $\exists x, [Q(x) \wedge R(x)]$ (d) $\exists x [Q(x) \wedge \sim R(x)]$

2.3 COMPOUND PROPOSITION

A proposition consisting of only a single propositional variable or a single propositional constant is called primary or primitive proposition or simply proposition. They can not be further subdivided. A proposition obtained from the combination of two more proposition by means of logical operators or connectives of two or ore proposition is called compound proposition.

2.3.1 Connectives

Any two statements may be combined in various ways to form new statement. To form new statement, the words, which are used are called connectives. We shall now discuss three most basic and fundamental connectives.These are negation, conjunction and disjunction, which are associated with English words 'Not', 'And', 'or' respectively. Some other connectives will be discussed subsequently. These connectives are shown below :

English Words	Name of Connective	Symbols	Order
Not	Negation	$\sim$ or $\urcorner$	1
And	Conjunction	$\wedge$	2
Or	Disjunction	$\vee$	3
One way implication	Conditional	$\Rightarrow$	4
If and only if (iff) (Two way implication)	Bi-conditional	$\Leftrightarrow$	5

2.3.2 Negation

Let p be a symbol for any statement, then negation of p is denoted by $\sim p$ or $\rceil p$ or p'. Let us consider the statement

$p = I$ went to my office yesterday.

Then negation of p is represented by

$\rceil p = I$ did not go to my office yesterday.

2.3.3 Conjunction

Let us consider two statements p and q given by

p = Ice is cold, and q = Blood is green.

These statements may be combined by the connective 'And' to form the new statement is given by

$p \wedge q$ = Ice is cold and blood is green.

This new statement is known as the conjunction of p and q. In general, we define the conjunction of p and q for arbitrary statements p and q to be the statement "both p and q" . In wording the word both is often omitted. From this new statement, we observe that the statement is true if both p and q are true, and false, if either one or both of p and q are false.

2.3.4 Disjunction

The two statements p and q *given in above section may also be combined in another way. Which is as follows :*

Either ice is cold or blood is green.

This new statement is referred to as the disjunction of p and q. This disjunction of p and q is denoted by $p \vee q$. The use of "either...or ..." in English is indistinct so in some usages imply "either...or ... or both," but is some other usages imply "either ... or ...", but not both. Let us consider an example :

This creature is either is dog or an animal;

The baby is either a boy or a girl.

The first statement of these is called inclusive disjunction because this allows the possibility that both may happen.The second statement says that both statements can not happen together. Thus this types of disjunction is called exclusive disjunction. The words "or both" are usually omitted, and the word either may be omitted if there is no in distinction.The disjunction of p and q or both are true and false only when both p and q are false.

2.3.5 Conditional

Let us suppose p and q are two statements. Then the statement " $p \Rightarrow q$ " which is read as if "p then q" or "p implies q" is called conditional statements. Here 'p' is called antecedent and 'q' is called consequent. The conditional statement '$p \Rightarrow q$' has four possibilities depends on p and q.

Case I. If p is true, q is false, then "$p \Rightarrow q$" is false.

Case II. If p is true, q is true, then "$p \Rightarrow q$" is true

Case II. If p is false, q is true, then "$p \Rightarrow q$" is true.

Case IV. If p is false, q is false, then "$p \Rightarrow q$" is true.

Alternative Wording for 'Conditional Statement'

- $p \Rightarrow q$ *i.e.*, p implies q
  This can also be expressed in passive voice "q is implied by p"
- "whenever p, we have q". Also q whenever p"
- "p is sufficient for q". Also, "p is sufficient condition for q".
- In order for q is hold, it is enough that we have p".
- "q is necessary for p".
- "p, only if q", *i.e.*, p can happen only if q happens as well.

Meaning of symbols

(i) $p \Rightarrow q$. The arrow symbol $\Rightarrow$ is pronounced "implies"
(ii) $q \Leftarrow p$. The arrow $\Leftarrow$ is pronounced "is implied by".
If the statement "p if and only if q" is true we have the following table :

Condition p	Condition q	$\Rightarrow$
True	True	Possible
True	False	Impossible
False	True	Impossible
False	False	Possible

From above table it is clear that for condition p to be true while q is false because $p \Rightarrow q$. Similarly, it is impossible for condition q to be true while p is false because $q \Rightarrow p$. Hence the two conditions p and q must be both true or both false.

2.3.6 Bi-Conditional

Let p and q be two statements, then $p \Leftrightarrow q$ is called bi-conditional statement. This statement can also be written as $(p \Rightarrow q)$ and $(q \Rightarrow p)$ *i.e.* $(p \Rightarrow q) \wedge (q \Rightarrow p)$. This bi-conditional statement is true only if both the statement p and q have same truth values.

Alternative Wording of Biconditional statement (p iff q)

- "p is necessary and sufficient condition for q"
- "p is equivalent to q".
- "$p \Leftrightarrow q$"
- The symbol $\Leftrightarrow$ is an amalgamation of the symbol $\Rightarrow$ and $\Leftarrow$.

☛ The mathematical usage of **and** and **not** corresponds closely with standard English. The use of **or** however does not. In standard English or often suggests a choice of one option or the other but not both. But mathematical or allows the possibility of both. The statement "p or q" means that p is true or q is true or both p and q are true.

2.3.7 Joint Denial

The word "NOR" is a combination of NOT and OR where NOT and OR stands for negation and disjunction. Let p and q be two statements. The "$p \downarrow q$" is called Joint Denial or "NOR" statement and read as "Neither p nor q". This "$p \downarrow q$" can also be written as

$$p \downarrow q \Leftrightarrow \rceil (p \vee q)$$

The Joint Denial statement $p \downarrow q$ is true only when p and q both are false.

2.3.8 NAND Statement

The word "NAND" is a combination of NOT and AND where NOT and AND stand for negation and conjunction respectively. Then the statement "$p \uparrow q$" is called NAND statement and this can be written as

$$p \uparrow q \Leftrightarrow \urcorner (p \wedge q)$$

This statement is false only if both p and q are true.

2.3.9 Types of Conditional Statement

Let p and q be any two statements, then there are some other conditional statements which are related to the conditional $p \Rightarrow q$.

(i) Converse Implication : The statement $q \Rightarrow p$ is called converse implication of the statement of $p \Rightarrow q$.

(ii) Inverse Implication : The statement $\urcorner p \Rightarrow \urcorner q$ is called inverse implication of the statement $p \Rightarrow q$.

(iii) Contrapositive Implication : The statement $\urcorner q \Rightarrow \urcorner p$ is called contrapositive of the statement $p \Rightarrow q$.

2.3.10 Use of Brackets

The use of brackets in the statements has an important role. By using brackets in the statements, the meaning or explanation of the statements are completely different. There are some rules related to the brackets.

(i) If the negation (*i.e.*, $\urcorner$) is repeated in the statement then there is no need of bracket.

For Example : $\urcorner \urcorner p$ and $(\urcorner (\urcorner p))$ both have same meaning.

(ii) If in a statement, the connectives of same type are present, then brackets are applied from the left. For example : $p \vee q \vee r \vee s = [\{(p \vee q) \vee r\} \vee s]$.

(iii) If the different connectives are used in the statement, then we remove the bracket of lower order connective, but we can not remove the bracket of higher order connective.

For example : $p \Rightarrow (q \wedge r) = p \Rightarrow q \wedge r$.

Here order of $\wedge$ is less than the order of $\Rightarrow$. Hence bracket is removed. Let us consider another example

$$p \vee (q \Rightarrow r) \neq p \vee q \Rightarrow r$$

In this example, bracket can not be removed.

2.4 TYPES OF SENTENCES

There are two type of sentences or statements.

(i) Simple Sentence : A statement which has no connectives is called simple sentence or simple statement (or Atomic statement).

For example : He is a boy or she is a girl. Both are simple.

(ii) Compound Sentence : A statement which is formed by two simple statement through the connective is called compound sentence or molecular sentence.

For Example :

(a) If you work hard, then you will get success.

(b) Suresh will play or he will leave ground.

2.4.1 Statement Form

This is a form obtained by simple statements, using finite number of connectives, is called statement form.

For example : Let p, q and r be three simple statements, then the statement $(p \wedge q) \Rightarrow r$ is a statement form. The connective which is used at the end of the statement form is called principal connective.

2.4.2 Principal connective

Thus in the above example, $\Rightarrow$ is the principal connective. The statements on both sides of the principal connective are called **Arguments.** Therefore, in the above example, the statement $(p \wedge q)$ and r are arguments. The statement form is also known as **Truth function.**

2.5 USE OF VENN DIAGRAMS IN CHECKING TRUTH AND FALSITY OF STATEMENTS

In this section, we shall discuss the usage of Venn diagrams to represent truth and falsity of statement of propositions.

ILLUSTRATIVE EXAMPLES

1. *Give the venn diagram for the truth of the following statement "Equivalent triangles are isosceles triangles.)*

Solution. Let U, E and S denotes the set of triangles in a plane, the set of equilateral triangles in a plane and the set of all isosceles triangles in a plane respectively.

Then, $E \subset U$ and $S \subset U$

It is also clear that $E \subset S$. Hence, we have the following Venn diagram to represent the truth of the given statement.

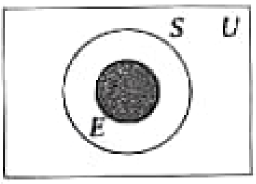

Fig. (1)

2. *Let U, P and T denotes respectively the set of human beings, the set of policemen and the set of all thieves. Write the truth value of the following statements from the Venn diagram given below :*

(i) No policeman is a thief.

(ii) Thieves are not policeman.

(iii) Men who are not policemen are thieves.

(iv) Some policemen are thieves.

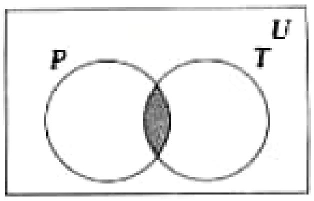

Fig. (2)

Solution. (i) From Venn diagram, it is clear that the policemen x is a thief also. Therefore, the given statement is not true. Hence, the truth value is F.

(ii) We find that $P \cap T \neq \phi$, therefore, there are some thieves who are also policemen. Hence, the statement is not true and the truth value is F.

(iii) From Venn diagram, it is clear that there are some human beings who are neither policemen nor thieves. Hence, the above statement is not true and the truth value is F.

(iv) Here, it is clear that the policemen x is a thief also. Hence, the given statement is true and the truth value is T.

3. *Use Venn diagrams to check the validity of the following argument*

S_1 : *If a man is a bachelor, he is unhappy.*

S_2 : *If a man is unhappy, he dies young.*

... ...

S : *All bachelors die young.*

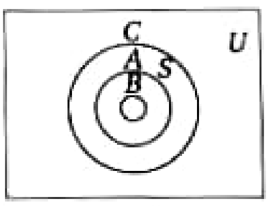

Fig. (3)

Solution. Define the following sets

A = set of all unhappy men

B = set of all bachelors

and C = set of all men who die young.

Then, the truth of the hypothesis S_1 is represented by placing the B inside the set A, *i.e.*, $B \subset A$.

Now, the truth of statement S_2 is represented by placing the set A completely inside the set C, *i.e.*, $A \subset C$.

Now, S_1 and S_2 are true

$\Rightarrow$ $B \subset A$ and $A \subset C$.

$\Rightarrow$ $B \subset C$, *i.e.*, all bachelors die young.

$\therefore$ S is true.

Hence, the given argument is a valid argument.

Exercise 2.1

Use Venn diagram to examine the validity of the following arguments :

1. (i) S_1 : Natural numbers are integers.
   S_2 : x is an integer
   ... ...
   S : x is a natural integer.
   (ii) S_1 : Natural numbers are integers.
   S_2 : x is an integer
   ... ...
   S : x is not a natural integer.
2. S_1 : All squares are rectangles.
   S_2 : x is not a rectangle.
   ... ...
   S : x is not a square.
3. S_1 : If it rains, Rijuta will be sick.
   S_2 : It did not rain.
   ... ...
   S : Rijuta was not sick.
4. S_1 : If 7 is less than 4, then 7 is not a prime number.
   S_2 : 7 is not less than 4.
   ... ...
   S : 7 is a prime number.
5. S_1 : All graduates get a job.
   S_2 : Rijuta is a graduate.
   ... ...
   S : Rajuta gets a job.

ANSWERS

1.(i) invalid, (ii) invalid; **2.** valid; **3.** valid; **4.** invalid; **5.** valid.

2.6 TRUTH VALUES AND TRUTH TABLE

Since we know that every statement has a unique value. This unique value is called **Truth value.** These truth values are 'True' or 'False'. Hence the truth value of every statement form is obtained by the truth values of its components. The range of statement form (or Truth function) is $\{T, F\}$. Hence we can say that there will be 2^n truth function having n statements. T and F represent the truth value, True and False respectively. These truth values are also represented by 1 and 0.

To show that the set of statement and the operations of conjunction, disjunction and negation, it is necessary first to define them equal. Let us consider two statement forms 'f' and 'g' having two component statements p and q each. These two statement forms are said to be equal if both have same truth values for each of the four possible ways.

(i) p false and q true.

(ii) p true and q false.

(iii) p and q both are true.

(iv) p and q both are false.

If in anyone of above possible ways, the truth values differ, then f and g are not equal. To understand the meaning and uses of different types of connectives, we analyse the truth values with the help of a table. This table is called truth table.

A truth able displays the relationship between the truth value of the proposition truth tables are especially valuable in the determination of the truth values of propositions from simple proposition.

2.6.1 Truth Table for Negation

Let p be any statement, then $\sim p$ is negation of p. The truth value of negation of p is opposite of truth value of p. Suppose p = Delhi is a capital of India.

Then, $\sim p$ = Delhi is not a capital of India.

The truth table is given below :

p	$\sim P(\neg p)$
T	F
F	T

Or

p	$\sim P(\neg p)$
1	0
0	1

2.6.2 Truth Table for Conjunction

Let p and q be two simple statements, then $p \wedge q$ is conjunction of p and q. Thus $p \wedge q$ has a true value if both p and q have true values. Let us consider

p = Lucknow is a capital of Uttar Pradesh; q = Delhi is a capital of India.

Then $p \wedge q$ means that Lucknow is a capital of Uttar Pradesh and Delhi is a capital of India. The truth table is given below :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Or

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

2.6.3 Truth Table for Disjunction

Let p and q be any two simple statements, then $p \vee q$ is called disjunction of p and q which is read as p or q. Let us consider

p = Mathematics is very hard subject; q = Commerce is very easy subject.

Then $p \vee q$ can be written as

$p \vee q$ = Mathematics is very hard subject or commerce is very easy subject. The statement $p \vee q$ has the truth value false only if both are false. Thus truth table is given below :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Or

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

2.6.4 Truth Table for conditional

Let p and q be two simple statements, then $p \Rightarrow q$ is the statement which is conditional. This statement has the truth value false only if antecedent p is true and consequent q is false. The truth table for $p \Rightarrow q$ is given below :

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Or

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

2.6.5 Truth Table for Bi-Conditional

Let p and p be any two simple statements. The statement of the type $p \Leftrightarrow q$ is called bi-conditional. This can also be written as $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$. This statement has the truth value true only if both p and q have same truth values.

Thus the truth table is given below :

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Or

p	q	$p \Leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

2.6.6 Truth table for Joint Denial

Let p and q be two statements. The statement $p \downarrow q$ is called Joint Denial statement which is read as 'Neither p nor q'. The statement $p \downarrow q$ has truth value 'True' when both p and q are false. Thus the truth table for $p \downarrow q$ is given below :

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Or

p	q	$p \downarrow q$
1	1	0
1	0	0
0	1	0
0	0	1

2.6.7 Truth Table for NAND Statement

Let p and q be two statements. The statement $p \uparrow q$ is called NAND statement. This statement is false only when both p and q are true. The truth table for NAND statement is given below :

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Or

p	q	$p \uparrow q$
1	1	0
1	0	1
0	1	1
0	0	1

2.6.8 Table for Symbols

To understand and then to write any statement into symbols there is a table given below :

(i)	If p then q	$p \Rightarrow q$
(ii)	p if q	$q \Rightarrow p$
(iii)	p only if q	$q \Rightarrow p$
(iv)	p unless q	$\neg q \Rightarrow p$
(v)	p is a sufficient condition for q	$p \Rightarrow q$
(vi)	p is a necessary condition for q	$q \Rightarrow p$
(vii)	A sufficient condition for p is q	$q \Rightarrow p$
(viii)	A necessary condition for p is q	$p \Rightarrow q$
(ix)	In order that p is sufficient that q	$q \Rightarrow p$

(x)	In order that p is necessary that q	$p \Rightarrow q$
(xi)	p if and only if q	$p \Leftrightarrow q$
(xii)	p is a necessary and sufficient condition for q	$p \Leftrightarrow q$

2.6.9 Precedence of logical operators

Rule (1) : Use parentheses to specify the order in which logical operators in a compound proposition are to be applied.

Rule (2) : To reduce the number of parentheses, we specify that the negation operator is applied before all the other logical operators.

Rule (3) : The conjunction operator takes precedence over the disjunction operator (we will continue to use parenthesis) so that the order of disjunction and conjunction operators is clear.

Rule (4) : The conditional and biconditional operator have lower precedence than the conjunction and disconjunction operators $\wedge$ and $\vee$. Consequently, $p \vee q \Rightarrow r$ is the same as $(p \vee q) \Rightarrow r$

Table : Precedence of logical operators

Operator	Precedence
$\rceil$ or $-$	1
$\wedge$ $\vee$	2 3
$\Rightarrow$ $\Leftrightarrow$	4 5

- We will use parenthesis when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator.

ILLUSTRATIVE EXAMPLES

1. *Which of the following expression are statements :*

(i) $1 + 2 + 3 = 1 \times 2 \times 3$

(ii) $\{2, 3\} \subset \{2, 4, 6\}$

(iii) *May you live long.*

(iv) *7 is a prime number*

(v) $5 \in \{1, 4, 5\}$

(vi) *All roses are white*

(vii) *What is your Name ?*

(viii) *The girls are beautiful.*

(ix) *Go to your home.*

(x) *Blood is red.*

Solution.

(i) Since $1 + 2 + 3 = 1 \times 2 \times 3$ or $6 = 6$.
Thus (i) is true. Hence, (i) is a statement.

(ii) $\{2, 3\} \subset \{2, 4, 6\}$.
This expression is false. Hence, it is a statement.

(iii) "May you live long" is not a declarative sentence. Hence, it is not a statement.

(iv) "7 is a prime number" which is declarative sentence having truth value 'True'.
Hence, it is a statement.

(v) $5 \in \{1, 4, 5\}$ is true. Hence, it is a statement.

(vi) "All roses are white". This is a declarative sentence and having truth value 'False'.
Hence, it is a statement.

(vii) "What is your name?" It is not a declarative sentence. Hence, it is not a statement.

(viii) "The girls are beautiful" This sentence is not declarative. Hence, it is not a statement.

(ix) "Go to your home". This is not a declarative sentence. Hence, it is not a statement.

(x) "Blood is red." It is a declarative sentence. Its truth value is True. Hence, it is a statement.

2. *Which of the following sentences are propositions ? What are the truth values of those that are propositions (statement) ?*

(i) Do you speak Hindi ? *(ii) Four is even.*

(iii) Please submit your proposal as soon as possible.

(iv) Do you speak English ? *(v)* $4 - x = 8$.

(vi) Please try to solve the problem.

Solution. We know that a statement is a declarative sentence which is either true or false, but not both. These two values are 'true' and 'false' denoted by symbolical T and F. Thus

(i) not statement (ii) statement (iii) not statement

(iv) not statement (v) not statement (vi) not statement

3. *If* $p \equiv$ *He is a carpenter and* $q =$ *He is making a table.*

Then write down the following statement into symbols :

(i) He is a carpenter and making a table.

(ii) He is a carpenter but is not making a table.

(iii) It is false that he is a carpenter or making a table.

(iv) Neither he is a carpenter nor he is making a table.

(v) He is not a carpenter and he is making a table.

(vi) It is false that he is not a carpenter or is not making a table.

(vii) He is a carpenter or making a table.

Solution. The solution of above compound statements in terms of p and q are given below :

(i) $p \wedge q$ (ii) $p \wedge \rceil q$ (iii) $\rceil (p \vee q)$ (iv) $\rceil p \wedge \rceil q$

(v) $\rceil p \wedge q$ (vi) $\rceil(\rceil p \vee \rceil q)$ (vii) $p \vee q$

4. *Consider the following :*

p *: This computer is good.*

q *: This computer is cheap.*

Write each of the following statements in symbolic forms :

(i) This computer is good and cheap. *(ii) This computer is not good but cheap.*

(iii) This computer is costly but good. *(iv) This computer is neither good nor cheap.*

(v) This computer is good or cheap.

Solution. (i) $p \wedge q$ (ii) $p' \wedge q$ (iii) $p \wedge q'$

(iv) $p' \vee q'$ (v) $p \vee q$.

5. *Consider the following :*

p *: Question paper is hard.*

p *: I will fail in the examination.*

Then translate the following sentences into symbols :

(i) Question paper is hard then I will fail in the examination.

(ii) If I will not fail in the examination, then question paper is not hard.

(iii) Question paper is not hard if and only if I will fail in the examination.

(iv) If question paper is not hard then I will pass in the examination.

Solution. (i) $p \Rightarrow q$ (ii) $\rceil p \Rightarrow \rceil q$ (iii) $\rceil p \Leftrightarrow q$ (iv) $\rceil p \Rightarrow \rceil q$

6. *Write the following in symbols :*

(i) Sachin will go out of station or will remain in his house and he will repair his radio.

(ii) *The necessary and sufficient condition for an infinite series Σu_n to be convergent is that limit of u_n as n tending to infinity must be zero.*
(iii) *We shall go to Delhi, but we shall not see the Red Fort.*
(iv) *Not only the children, but also Mothers and Fathers were killed.*
(v) *If teams do not arrive or the weather is bad, then there will be no match.*

Solution.

(i) Let $p \equiv$ Sachin will go out of station.
$q \equiv$ Sachin will remain in his house.
$r \equiv$ Sachin will repair his radio.
Thus the statement has the symbol $p \vee (q \wedge r)$.

(ii) $p \equiv$ An infinite series Σu_n to be convergent.
$q \equiv$ limit of u_n must be zero as n tending to infinity.
Thus the statement has the symbol $p \Leftrightarrow q$.

(iii) $p \equiv$ We shall go to Delhi.
$q \equiv$ We shall not see the Red Fort.
Thus the statement is $p \wedge \rceil q$.

(iv) $p \equiv$ children were killed.
$q \equiv$ Mothers were killed.
$r \equiv$ Fathers were killed.
Thus the statement is $p \wedge (q \wedge r)$.

(v) $p \equiv$ The teams do not arrive.
$q \equiv$ Whether is bad.
$r \equiv$ there will be no match.
Thus the statement is $(p \vee q) \Rightarrow r$.

7. *If p = Ramesh is a player, q ≡ Mohan is an intelligent boy, then write the following symbols into sentences :*

(i) $p \wedge q$ (ii) $\rceil p \wedge \rceil q$ (iii) $p \wedge \rceil q$ (iv) $\rceil (p \wedge q)$
(v) $\rceil p \Leftrightarrow q$ (vi) $p \Rightarrow \rceil q$

Solution. (i) Ramesh is a player and Mohan is an intelligent boy.
(ii) Neither Ramesh is a player nor Mohan is an intelligent boy.
(iii) Ramesh is a player and Mohan is not an intelligent boy.
(iv) It is false that Ramesh is a player and Mohan is an intelligent boy.
(v) Ramesh is not a player if and only if Mohan is an intelligent boy.
(vi) If Ramesh is a player, then Mohan is not an intelligent boy.

8. *If p ≡ Money is evil, q ≡ Wise men are poor, r ≡ beggars are failures. Then translate each of the following statements into symbols :*

(i) *Wise men are poor only if money is evil.*
(ii) *Money is evil unless wise men are poor.*
(iii) *That beggars are failures is a sufficient condition that money is evil.*
(iv) *A necessary condition for money to be evil is that beggars are failures.*
(v) *Money is evil and beggars are failures if wise men are poor.*
(vi) *Unless beggars are failures, wise men are not poor and money is not evil.*

Solution. (i) $p \Rightarrow q$ (ii) $\rceil q \Rightarrow p$
(iii) $r \Rightarrow p$ (iv) $p \Rightarrow r$ (v) $q \Rightarrow (p \wedge r)$ (vi) $\rceil r \Rightarrow \rceil p \wedge \rceil q$.

9. *Write the negation of each of the following statements in terms of symbols :*

(i) *It will rain unless the barometer rises.*
(ii) *I grow fat only if I eat too much.*
(iii) *A necessary condition that two triangles are equivalent is that they have the same area.*
(iv) *In order to live well, it is sufficient to be wealthy.*

Solution. (i) Let $p \equiv$ It will rain, $q \equiv$ Barometer rises.

Then the statement (i) can be written as $\neg q \Rightarrow p$.

Thus the negation is $\neg(\neg q \Rightarrow p)$.

(ii) Let $p \equiv$ I grow fat, and $q \equiv$ I eat too much.

Then the statement (ii) can be written as $q \Rightarrow p$.

Thus its negation is $\neg(q \Rightarrow p)$

(iii) Let $p \equiv$ Two triangles are equivalent, and $q \equiv$ They have the same area.

Then the statement (iii) can be written as $p \Rightarrow q$.

Thus its negation is $\neg(p \Rightarrow q)$

(iv) Let $p \equiv$ Live well, and $q \equiv$ To be wealthy.

Then the statement (iv) can be put as $q \Rightarrow p$.

Thus its negation is $\neg(q \Rightarrow p)$.

10. *Write the negation of the following :*

(i) If she studies, she will pass in exam.

(ii) If it rains, then they will not go for picnic.

(iii) Every even integer greater than 4 is the sum of two primes.

(iv) Some people have no scooter.

(v) No one wants to buy my house.

Solution. (i) If she will not study, she will not pass in exam.

(ii) If it will not rain, then they will go for picnic.

(iii) Every odd integer greater than 4 is the sum of two primes.

(iv) Some people have scooter.

(v) Everyone wants to buy my house.

11. *Write the negation of the following :*

(i) Anil is not rich and Kanchan is poor.

(ii) A cow is an animal.

(iii) If the determinant of a system of linear equations is zero, then either the system has no solution or it has an infinite number of solution.

Solution.

(i) Anil is rich and Kanchan is not poor.

(ii) A cow is not an animal.

(iii) p : Determinant of a system of linear equation is zero.

q : System has no solution.

r : System has an infinite number of solution.

Its negation is $\neg p \rightarrow \neg q \vee \neg r$.

12. *Write in words the converse, inverse, contrapositive and negation of the implication "If 2 is less then 3, then 1/3 is less than 1/2.*

Solution. Let $p \equiv 2$ is less than 3. $q \equiv 1/3$ is less than 1/2.

Then implication is $p \Rightarrow q$:

(i) Converse of $p \Rightarrow q$ is $q \Rightarrow p$. In words $q \Rightarrow p$ means "If 1/3 is less than 1/2, then 2 is less than 3".

(ii) Inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$. Thus in words, $\neg p \Rightarrow \neg q$ means "If 2 is not less than 3, then 1/3 is not less than 1/2".

(iii) Contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. Thus in words $\neg q \Rightarrow \neg p$ means "If 1/3 is not less than 1/2, then 2 is not less than 3".

(iv) Negation of $p \Rightarrow q$ is $\neg(p \Rightarrow q)$. Thus in words $\neg(p \Rightarrow q)$ means "It is false than p implies q".

13. *Let p be a statement "Eight is an even number", and let q be a statement "Candy is sweet". Write in words (i) the implication $p \Rightarrow q$, (ii) its converse, (iii) its inverse, (iv) its contrapositive, (v) its negation.*

Solution. (i) $p \Rightarrow q$ means "If eight is an even number, then candy is sweet."

(ii) Converse of $p \Rightarrow q$ is $q \Rightarrow p$ it means "If candy is sweet, then eight is an even number."

(iii) Inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$. It means "If eight is not an even number, then candy is not sweet".

(iv) Contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. It means "If candy is not sweet, then eight is odd number."

(v) Negation of $p \Rightarrow q$ is $\neg(p \Rightarrow q)$. In words, we can write "It is false that p implies q" or "Eight is an even number, and candy is not sweet".

☛ $\neg(p \Rightarrow q) = \neg(\neg p \vee q) = p \wedge \neg q$.

14. *If $p \equiv$ Missiles are costly and $q \equiv$ Grandma chews gum. Write in words, the following statement given in symbol :*

(i) $p \vee \neg q$ *(ii)* $\neg p \wedge \neg q$ *(iii)* $(p \wedge \neg q) \vee (\neg p \wedge q)$

Solution. (i) $p \vee \neg q \equiv$ Either missiles are costly or Grandma does not chew gum.

(ii) $\neg p \wedge \neg q \equiv$ Missiles are not costly and Grandma does not chew gum.

(iii) Either missiles are costly and Grandma does not chew gum, or missiles are not costly and Grandma chews gum.

15. *If $p \equiv$ Mathematics is easy and $q \equiv$ Two is less than three. Write in words the following statement given in symbols :*

(i) $\neg(p \wedge q)$ *(ii)* $\neg(p \vee q)$ *(iii)* $\neg p \vee q$ *(iv)* $(p \wedge \neg q) \vee (\neg p \wedge q)$

Solution.

(i) It is false that mathematics is easy and two is less than three.

(ii) It is false that either mathematics is easy or two is less than three.

(iii) Either mathematics is not easy or two is not less than three.

(iv) Either mathematics is easy and two is less than three or mathematics is not easy and two is less than three.

16. *If $p \equiv$ It is 10 o' clock, $q \equiv$ the train is late, then state in words the following resultants :*

(i) $q \vee \neg p$ *(ii)* $\neg p \wedge q$ *(iii)* $p \wedge \neg q$ *(iv)* $\neg(p \wedge q) = \neg p \vee \neg q$

(v) $\neg p \wedge \neg q$

Solution.

(i) $q \vee \neg p$: The train is late or it is not 10 o' clock.

(ii) $\neg p \wedge q$: It is not 10 o' clock and the train is late.

(iii) $p \wedge \neg q$: It is 10 o' clock and the train is not late.

(iv) $\neg(p \wedge q) = \neg p \vee \neg q$: It is not 10 o' clock, or the train is not late.

(v) $\neg p \wedge \neg q$: It is not 10 o' clock and the train is not late.

17. *Consider the following :*

p : You take a course in Discrete Mathematics.

q : You understand logic.

r : You get an A grade on the final exam.

Write in simple sentences the meaning of the following :

(i) $q \Rightarrow r$ *(ii)* $\neg p \Rightarrow \neg q$ *(iii)* $(p \wedge q) \Rightarrow \neg r$ *(iv)* $(p \wedge q) \Rightarrow r$

Solution.

(i) If you understand logics, then you get an A grade in the final exam.

(ii) If you will not take a course in Discrete Mathematics, then you will not understand logic.

(iii) If you take a course in Discrete Mathematics and understand logic, then you may not get an A grade in the final exam.

(iv) If you take a course in Discrete Mathematics and you understand logic, then you get an A grade in the final exam.

18. *Construct a truth table for each of the following functions :*

(i) $(p \wedge q \wedge r) \vee (\rceil p \wedge q \wedge \rceil r) \vee (\rceil p \wedge \rceil q \wedge \rceil r)$

(ii) $(p \vee q \vee r) \wedge (\rceil p \vee q \vee \rceil r) \wedge (\rceil p \vee \rceil q \vee \rceil r)$

(iii) $\rceil(\rceil p \vee (q \wedge r) \wedge \{(p \wedge q) \vee (\rceil q \wedge r)\}$

Solution. (i) Truth table

p	q	r	$\rceil p$	$\rceil q$	$\rceil r$	$(p \wedge q \wedge r)$	$(\rceil p \wedge q \wedge \rceil r)$	$(\rceil p \wedge \rceil q \wedge \rceil r)$	(i)
T	T	T	F	F	F	T	F	F	T
T	T	F	F	F	T	F	F	F	F
T	F	T	F	T	F	F	F	F	F
F	T	T	T	F	F	F	F	F	F
T	F	F	F	T	T	F	F	F	F
F	T	F	T	F	T	F	T	F	T
F	F	T	T	T	F	F	F	F	F
F	F	F	T	T	T	F	F	T	T

(ii) Truth table

p	q	r	$\rceil p$	$\rceil q$	$\rceil r$	$(p \vee q \vee r)$	$(\rceil p \vee q \vee \rceil r)$	$(\rceil p \vee \rceil q \vee \rceil r)$	(ii)
T	T	T	F	F	F	T	T	F	F
T	T	F	F	F	T	T	T	T	T
T	F	T	F	T	F	T	F	T	F
F	T	T	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T	T	T
F	T	F	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T	T	T
F	F	F	T	T	T	F	T	T	F

(iii) Truth table

p	q	r	$\rceil p$	$\rceil q$	$p \wedge q$	$(q \wedge r)$	$\rceil q \wedge r$	$\rceil p \vee (q \wedge r)$	$\{\rceil p \vee (q \wedge r)\}$	$\{(p \wedge q) \vee (\rceil q \wedge r)\}$	(iii)
T	T	T	F	F	T	T	F	T	F	T	F
T	T	F	F	F	T	F	F	F	T	T	T
T	F	T	F	T	F	F	T	F	T	T	T
F	T	T	T	F	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F	F	T	F	F
F	T	F	T	F	F	F	F	T	F	F	F
F	F	T	T	T	F	F	T	T	F	T	F
F	F	F	T	T	F	F	F	T	F	F	F

19. *Construct the truth table of the following :*

(a) $(p \wedge q) \wedge (q \wedge r) \wedge (r \wedge s)$ (b) $\rceil(\rceil(p \wedge \rceil q)$

(c) $(\rceil p \wedge (\rceil q \wedge s)) \vee (r \wedge s) \wedge (s \vee r)$

Solution. (a)

p	q	r	s	$p \wedge q$	$q \wedge r$	$r \wedge s$	$(p \wedge q) \wedge (q \wedge r) \wedge (r \wedge s)$
T	T	T	T	T	T	T	T
T	T	T	F	T	T	F	F
T	T	F	T	T	F	F	F
T	F	T	T	F	F	T	F
F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	F
T	F	F	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	T	F
T	F	T	F	F	F	F	F
F	T	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	F	F	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	T	F	F	F	F
F	F	F	F	F	F	F	F

(b)

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

(c)

p	q	r	s	$\neg p$	$\neg q$	$\neg q \wedge s$	$\neg p \wedge (\neg q \wedge s)$ (1)	$r \wedge s$ (2)	$s \vee r$ (3)	(2) $\wedge$ (3) (4)	(1) $\vee$ (4)
T	T	T	T	F	F	F	F	T	T	T	T
T	T	T	F	F	F	F	F	F	T	F	F
T	T	F	T	F	F	F	F	F	T	F	F
T	F	T	T	F	T	T	F	T	T	T	T
F	T	T	T	T	F	F	F	T	T	T	T
T	T	F	F	F	F	F	F	F	F	F	F
T	F	F	T	F	T	T	F	F	T	F	F
T	F	T	F	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F	F	F	F	F
F	T	T	F	T	F	F	F	F	F	F	F
T	T	F	T	F	F	F	F	F	T	F	F

F	T	F	F	T	F	F	F	F	F	F	F
F	F	T	F	T	T	F	F	F	T	F	F
F	F	F	T	T	T	T	T	F	T	F	T
F	F	F	F	T	T	F	F	F	F	F	F

20. *Write the negation for the statement* $\forall \in R, x > 3 \Rightarrow x^2 > 9$

Solution. Let $P(x)$ and $Q(x)$ denote '$x > 3$' and $x^2 > 9$.

Then the given statement can be written as

$$\forall \; x\,(P(x) \Rightarrow Q(x))$$

The negation of this statement is

$$\exists \, x\,(P(x) \wedge -Q(x))$$

i.e., there exist a real number x such that $x \leq 3$ and $x^2 \leq 9$.

2.7 TAUTOLOGY

We have already discussed about the compound statements which formed with the help of simple statements using connectives. The truth values of this compound statement depend on the truth values of simple statement substituted for the variables. Thus the truth table of a resulting compound statement gives the summary of all its truth values for all possible choice of values of the variables. Therefore sometimes the truth values of the compound statement may, be "True (T)" and sometimes "False (F)". But there are some compound statements whose truth values are always T or always F irrespective of all possible truth values given to the variables.

Definition : *A statement whose truth value is always T (i.e., True) is called a "Tautology" and the statement whose truth value is always False (i. e., F) is called a "Contradiction".*

- Negation of a tautology is a contradiction while negation of contradiction is a tautology.

2.7.1 Logical Equivalence

The two compound statements are said to be logically equivalents if both have same truth values for all possible assignments given to the variables. This logically equivalent is also known as tautologically equivalent.

2.8 DUALITY

The two compound statements are said to be dual of each other if either one can be obtained from other by interchanging $\wedge$ and $\vee$ provided both remain valid.

For example : The dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$ and the dual of $\urcorner(p \vee q) \wedge \{p \vee \urcorner(q \wedge \urcorner s)\}$ is $\urcorner(p \wedge q) \vee \{p \wedge \urcorner(q \vee \urcorner s)\}$

2.9 ALGEBRA OF STATEMENTS

Now we shall discuss some tautological laws related to the statements under the algebra of statements. Under this section, we shall also discuss some laws which are tautologies. For simplicity let us take 't' for tautology and 'f' for contradiction.

2.9.1 Commutative Laws

(a) $(p \vee q) \Leftrightarrow (q \vee p)$ (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

Proof : (a) Truth table for $(p \vee q) \Leftrightarrow (q \vee p)$

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \Leftrightarrow (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Thus in the last column all the truth values are T. Hence it is a tautology.
(b) Truth table for $(p \wedge q) \Leftrightarrow (q \wedge p)$

p	q	$p \wedge q$	$q \wedge p$	$(p \wedge q) \Leftrightarrow (q \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

Similarly, $(p \wedge q) \Leftrightarrow (q \wedge p)$ is a tautology.

- The notation $\Leftrightarrow$ can be replaced by '$\equiv$' or '$=$'
  i.e., $(p \vee q) = (q \vee p)$; $(p \wedge q) = (q \wedge p)$

2.9.2 Associative Laws

(a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ (b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Proof : (a) Truth table for $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

p	q	r	$p \vee q$	$q \vee r$	$p \vee (q \vee r)$	$(p \vee q) \vee r$	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	F	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

Thus $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ is a tautology.
(b) Truth table for $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

p	q	r	$p \wedge q$	$q \wedge r$	$p \wedge (q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	F	T
F	T	T	F	T	F	F	T
T	F	F	F	F	F	F	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	T

Similarly, $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ is a tautology.

2.9.3 Distributive Laws

(a) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$(b) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Proof : (a) Truth table for $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$q \vee r$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	F	T	F	F	F	T
T	F	F	F	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

Thus the given expression has the same truth value T. Hence it is a tautology.

(b) Truth table for $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

p	q	r	$p \vee q$	$q \wedge r$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T	T
T	F	T	T	F	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	T	F	T	T	T	T
F	T	F	T	F	F	F	F	T
F	F	T	F	F	T	F	F	T
F	F	F	F	F	F	F	F	T

Similarly, $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ is a tautology

2.9.4 Idempotent Laws

(a) $(p \vee p) \Leftrightarrow p$ (b) $(p \wedge p) \Leftrightarrow p$.

Proof : (a) Truth table for $(p \vee p) \Leftrightarrow p$

p	p	$p \vee p$	$(p \vee p) \Leftrightarrow p$
T	T	T	T
F	F	F	T

Here, the last column has same truth value T. Hence it is a tautology.

(b) Truth table for $(p \wedge p) \Leftrightarrow p$

p	p	$p \wedge p$	$(p \wedge p) \Leftrightarrow p$
T	T	T	T
F	F	F	T

Similarly, $(p \wedge p) \Leftrightarrow p$ is a tautology.

2.9.5 Absorption Laws

(a) $p \vee (p \wedge q) \Leftrightarrow p$; (b) $p \wedge (p \vee q) \Leftrightarrow p$

Proof : (a) Truth table for $p \vee (p \wedge q) \Leftrightarrow p$

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee (p \wedge q) \Leftrightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

Since the last column contains all the truth values true. Hence $p \vee (p \wedge (p \wedge q) \Leftrightarrow p$ is a tautology.

(b) Truth table for $p \wedge (p \vee q) \Leftrightarrow p$

p	q	$p \vee q$	$p \wedge (p \vee q)$	$p \wedge (p \vee q) \Leftrightarrow p$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

Similarly, $p \wedge (p \vee q) \Leftrightarrow p$ is a tautology.

2.9.6 De-Morgan's Laws

(a) $\urcorner(p \vee q) \Leftrightarrow (\urcorner p \wedge \urcorner q)$; (b) $\urcorner(p \wedge q) \Leftrightarrow (\urcorner p \vee \urcorner q)$.

Proof : (a) Truth table for $\urcorner(p \vee q) \Leftrightarrow (\urcorner p \wedge \urcorner q)$

p	q	$p \vee q$	$\urcorner p$	$\urcorner q$	$\urcorner(p \vee q)$	$(\urcorner p \wedge \urcorner q)$	$\urcorner(p \vee q) \Leftrightarrow (\urcorner p \wedge \urcorner q)$
T	T	T	F	F	F	F	T
T	F	T	F	T	F	F	T
F	T	T	T	F	F	F	T
F	F	F	T	T	T	T	T

The last column contains all the truth values 'true'. Hence $\urcorner(p \vee q) \Leftrightarrow (\urcorner p \wedge \urcorner q)$ is a tautology.

(b) Truth table for $\urcorner(p \wedge q) \Leftrightarrow (\urcorner p \vee \urcorner q)$

p	q	$\urcorner p$	$\urcorner q$	$(p \wedge q)$	$\urcorner(p \wedge q)$	$(\urcorner p \vee \urcorner q)$	$\urcorner(p \wedge q) \Leftrightarrow (\urcorner p \vee \urcorner q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Similarly, $\urcorner(p \wedge q) \Leftrightarrow (\urcorner p \vee \urcorner q)$ is a tautology.

2.9.7 Detachment Law

$$[(p \Rightarrow q) \wedge p] \Rightarrow q.$$

Proof : Truth table for $[(p \Rightarrow q) \wedge p] \Rightarrow q$

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The last column contains all the truth values 'True'. Hence it is a tautology.

2.9.8 Chain Laws

$$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$$

Proof : Truth table for chain law

p	q	r	$(p \Rightarrow q)$	$(q \Rightarrow r)$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$(p \Rightarrow r)$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the last column contains all the truth values 'true'. Hence it is a tautology.

- The chain law can be restated as follows "If p implies q and q implies r, then p implies r".

2.9.9 Identity Laws

If t stands for tautology and f stands for contradiction. Then

(a) $p \wedge t = t \wedge p = p$ (b) $p \vee f = f \vee p = p$.

Proof : Truth table for (a)

p	t	$p \wedge t$	$t \wedge p$
T	T	T	T
F	T	F	F

Truth table for (b)

p	f	$p \vee f$	$f \vee p$
T	F	T	T
F	F	F	F

2.9.10 Complement laws

(a) $p \vee \rceil p = t$ (b) $p \wedge \rceil p = f$

Proof : Truth table for (a)

p	t	$\rceil p$	$p \vee \rceil p$
T	T	F	T
F	T	T	T

Truth table for (b)

p	f	$\rceil p$	$p \wedge \rceil p$
T	F	F	F
F		T	F

2.9.11 Functionally Complete set of operations

"A set of operations or connectives is said to be functionally complete if every statement can be expressed entirely in terms of the operations in the set.

Illustrations :

(1) We know that $p \Rightarrow q$ is equal to $\sim pvq$ therefore it is possible to replace each occurrence of $\Rightarrow$ in any statement with an equivalent expression involving $\sim$ and $\vee$.

(2) We know that $p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p) = (\sim p \vee q) \wedge (\sim q \vee p)$

Therefore the symbol $\Leftrightarrow$ can be replaced by connectives $\vee$, $\wedge$ and $\sim$.

(3) We know that (by Demorgan's law)

$\sim(p \wedge q) = \sim p \vee \sim q$ or $p \wedge q = \sim(\sim p \vee \sim q)$

Similarly $p \vee q = \sim(\sim p \wedge \sim q)$

Hence, it is possible to replace $\wedge$ in any statement by connectives $\sim$ and $\vee$. So. any statement can be expressed in an equivalent statement containing $\sim$ and $\wedge$ only, which shows that $\{\wedge, \sim\}$ is functionally complete set of operations. In a similar in an equivalent statement containing $\vee$ and $\sim$ only.

Hence, $\{\vee, \sim\}$ is also functionally complete set.

ILLUSTRATIVE EXAMPLES

1. *Show that $\{\Rightarrow, \sim\}$ is a functionally complete set.*

Solution : From above illustrations we know that $\{\vee, \sim\}$ is a functionally complete set. Therefore, any statement can be expressed in an equivelent statement containing $\vee$ and $\sim$

Since $p \vee q = (\sim p) \Rightarrow q$

$\therefore$ We an replace the connectives $\vee$ in ay statement by $\sim$ and $\Rightarrow$. So, given set $\{\Rightarrow, \sim\}$ is a functionally complete set.

2. *Show that $\{\wedge, \vee\}$ is not a functionally complete set.*

Solution : We know that connectives '$\sim$' can not be expressed entirely in terms of $\vee$ and $\wedge$, therefore, any statement containing '$\sim$' can not be expressed in an equivalent statement containing $\wedge$ and $\vee$ only. Hence, the given set $\{\wedge, \vee\}$ is not a functionally complete set.

2.10 VALIDITY OF ARGUMENTS : METHOD OF PROOFS

The main problem related to the symbolic logic is the investigation of the process of reasoning. We know that in every deductive science, there are no positive declarations having absolute truth. Therefore, we assumed a certain set of statements without proof, and from this set, we obtained some other statements using connectives. We now proceed to investigate those processes which will be accepted as valid in the derivation of a statement, called the conclusion, from other given statements, called premises.

An argument is a process by which a conclusion is obtained from given set of premises. Let $p_1, p_2, \ldots, p_n$ be the all premises and r is the conclusion. Then an argument which yields a conclusion r from premises $p_1, p_2 \cdots p_n$ is valid if and only if the statement $(p_1 \wedge p_2 \wedge \ldots, p_n) \Rightarrow r$ is a tautology. There are three methods to check the validity of a given argument.

Method I : By Truth Table : The first method is to check the validity directly from the truth table, that is for the argument $(p_1 \wedge p_2 \wedge \ldots, \wedge p_n) \Rightarrow r$ the truth table method is used to show that $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Rightarrow r$ is a tautology.

Method II : By Simplification Method : In this method we shall have to show that the statement $(p_1 \wedge p_2 \wedge .., \wedge p_n) \Rightarrow r$ can be reduced to 1, using the standard methods of simplification.

(*) **Method III : Using Rule of Inference :** This method is to reduce the given argument to a series of argument each of which is valid. This method is often the simplest of the three.

Two of the most frequently used valid arguments are the rule of detachment and the law of syllogism. The rule of detachment is given by

$$\begin{array}{c} p \\ \underline{p \Rightarrow q} \\ q \end{array}$$

Here p and $(p \Rightarrow q)$ are premises and the conclusion is written below a horizontal line, which is q as well. The law of syllogism is given by

$$\begin{array}{c} p \Rightarrow q \\ \underline{q \Rightarrow r} \\ p \Rightarrow r \end{array}$$

In checking the validity of any argument, we shall also assume that the following rules of substitution are permissible to use.

Rule I : Any valid argument which involves a statement variable will remain valid if every occurrence of a given variable is replaced by a specific statement.

Rule II : Any valid argument will remain valid if any equivalent statement occurrence of a statement is replaced by an equivalent statement.

☛ The validity of any argument does not depend on the truth values 'True' or 'False' of the conclusion. For example, Let us consider two arguments :

1. If ice is warm, then snow is black.
   Ice is warm
   Snow is black
2. 5 is an odd integer.
   If 4 is an even integer, then 5 is an odd integer
   4 is an even integer

Here the first is valid although the conclusion is false, while the second is invalid although the conclusion is true.

2.11 RULE OF DETACHMENT OR MODUS PONENS

The valid argument

$$\begin{array}{c} p \\ \underline{p \Rightarrow q} \\ q \end{array}$$

is known as rule of detachment. It is also known as modus ponens.

Law of Syllogism. The argument

$$\begin{array}{c} p \Rightarrow q \\ \underline{q \Rightarrow r} \\ p \Rightarrow r \end{array}$$

is valid argument and known as law of syllogism.

Its implication form is $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow [(p \Rightarrow r)]$

Truth table for law of syllogism

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From above table, it is clear that, the law of syllogism is a valid argument.

You must know

- The word *theorem* should not be confused with the word 'theory'. A *theorem* is a specific statement that can be proved. A theory is a broader assembly of ideas and a particular issue.
- The word *theorem* Carries the connotation of importance and generality. For example, Pythogoorean theorem certainly deserves to be called a theorem.
- Some words that are alternatives to *theorem* are given below.

(i) Result : A modest generic word for a theorem. Both important and unimportant theorem can be called results.

(ii) Fact : A very miner theorem. For example '2 +4 = 6' is a fact

(iii) Proposition : A minor theorem. A proposition is more important or more general than a fact but not as prestigious as a theorem.

(iv) Lemma : The lemma are the parts or tools, used to build the more complicated proof of the theorem. So , lemma is a theorem whose main purpose is to help prove another more important theorem.

(v) Corollary : A result with a short proof whose main step is the use of another previously proved theorem.

(vi) Claim : A claim is a theorem whose statement usually appears inside the proof of a theorem. The purpose of a claim is to help to organise key steps in the proof. It is very similar to lemma.

2.12 RULES OF INFERENCE

In the following table, we give some rules of inference. Here, observe that for valid argument, we can use the rules of inference :

Rules of Inference	Implication Form
RI_1 : Addition : $\dfrac{p}{\therefore p \vee q}$	$p \Rightarrow (p \vee q)$
RI_2 : Conjunction : $\dfrac{q}{\therefore p \wedge q}$	$q \Rightarrow p \wedge q$
RI_3 : Simplification : $\dfrac{p \wedge q}{\therefore p}$	$(p \wedge q) \Rightarrow p$

RI_4 : Modus ponens : p $p \Rightarrow q$ $\therefore q$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
RI_5 : Modus tollens : $\rceil q$ $p \Rightarrow q$ $\therefore \rceil p$	$(\rceil q \wedge (p \Rightarrow q)) \Rightarrow \rceil p$
RI_6 : Disjunctive syllogism : $\rceil p$ $p \vee q$ $\therefore q$	$(\rceil p \wedge (p \vee q)) \Rightarrow q$
RI_7 : Hypothetical syllogism : $p \Rightarrow q$ $q \Rightarrow r$ $\therefore p \Rightarrow r$	$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$
RI_8 : Constructive dilemma : $(p \Rightarrow q) \wedge (r \Rightarrow S)$ $p \vee r$ $\therefore q \vee S$	$(p \Rightarrow q) \wedge (r \Rightarrow S) \wedge (p \vee r) \Rightarrow (q \wedge S)$
RI_9 : Destructive dilemma : $(p \Rightarrow q) \wedge (r \Rightarrow S)$ $\rceil q \vee \rceil S$ $\therefore p \vee r$	$(p \Rightarrow q) \wedge (r \Rightarrow S) \wedge (\rceil q \vee \rceil S)$ $\Rightarrow (\rceil p \vee \rceil r)$

ILLUSTRATIVE EXAMPLES

1. *Check the validity of the following argument.*

$$\begin{array}{l} p \\ p \Rightarrow q \\ \underline{q \Rightarrow r} \\ r \end{array}$$

Solution : Method-I : For validity of the above argument, following statement

$$f = [p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow r$$

must be a tautology.

Truth table

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \wedge (p \Rightarrow q)$	$p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)$	f
T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	F	F	T

From above, it is clear that f is a tautology. Hence given argument is valid.

Method-II : *(Simplification method)*

$f = [p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow r$

$= -[(p \wedge (-p \vee q) \wedge (-q \vee r)] \vee r$ $\quad (\because\ p \Rightarrow q = \sim p \vee q)$

$= \sim p \vee (p \wedge -q) \vee (q \wedge -r) \vee r$ $\quad (\because\ -(p \wedge q) = \sim p \vee v - q)$

$= (-p \vee p) \wedge (\sim p \vee \sim q) \vee (q \wedge -r) \vee r$ (By distribulirity)

$= T \wedge (\sim p \vee -q) \vee r \vee (q \wedge \sim r)$ $\quad (\because -p \vee p = T)$

$= (\sim p \vee -q) \vee \{(v \vee q) \wedge (r \vee -r)]$ $\quad (\because p \wedge T = p)$

$= (\sim p \vee -q) \vee \{(r \vee q) \wedge T)$ $\quad (\because\ r \vee \sim r = T)$

$= \{\sim p \vee \sim q) \vee (r \vee q)$ $\quad (\because p \wedge T = p)$

$= \{(-p \vee \sim q) \vee q\} \vee r$ $\quad (\because p \vee q = q \vee p)$

$= \{\sim p \vee (\sim q \vee q)\} \vee r$ (By associalivity)

$= \{-p \vee T) \vee r$ $\quad (\because\ -p \vee p = T)$

$= T$

$\Rightarrow$ Hence, given argument is valid.

Method-III : *(Rule of inference)*

Consider following sequence of argument,

p	a premise
$p \Rightarrow q$	a premise
p	by modus ponens
$q \Rightarrow r$	a premise
r	by modus ponens

Hence, given argument is valid.

2. *Check the validity of the following argument*

(i) $p \Rightarrow q$
$\underline{r \Rightarrow \sim q}$
$p \Rightarrow \sim r$

(ii) p
$p \wedge q \Rightarrow r \vee s$
$\underline{\sim s}$
r

Solution : (1) Since the statement $r \Rightarrow \sim q$ is equal to $q \Rightarrow -r$ we can replace the premise $r \Rightarrow \sim q$ by $q \Rightarrow -r$

Now $P \Rightarrow q$
$\underline{q \Rightarrow \sim r}$
$p \Rightarrow \sim r$

is valid argument by the law of syllogism. Hence given argument is valid.

(2) We a take as premise for the indirect proof of all given premise except $\sim s$ and the negation of the conclusions $\sim r$. We shall show that the following argument is valid.

p
$p \wedge q \Rightarrow r \vee s$
q
$\underline{\sim r}$
s

Now,

p	a premise
q	a premise
$p \wedge q$	a conclusion because $p \wedge q \Rightarrow p \wedge q$ is always a tautology.

Further

$p \wedge q$	a valid conclusion
$p \wedge q \Rightarrow r \vee s$	a premise
$r \vee s$	a valid conclusion by modus ponens
$-r$	a premise
s	a valid Conclusion $(\because (r \vee s) \Rightarrow -r \Rightarrow s$ is a tautology)

3. *Check the validity of the following argument : If Shreesh has completed B.E. or M.B.A., then he is assured of a good job. If Shreesh is assured of a good job, he is happy. Shreesh is not happy. So shreesh has not completed M.B.A.*

Solution. We can name the proposition in the following way :

P denotes "Shreesh has completed B.E. "

Q denotes "Shreesh has completed M.B.A."

R denotes "Shreesh is assured of a good job"

S denotes "Shreesh is happy".

The given premises are :

(i) $(P \vee Q) \Rightarrow R$ (ii) $R \Rightarrow S$ (iii) $\rceil S$

The conclusion is $\rceil Q$

1.	$(P \vee Q) \Rightarrow R$	Premise (i)
2.	$R \Rightarrow S$	Premise (ii)
3.	$(P \vee Q) \Rightarrow S$	Hypothetical syllogism RI_7
4.	$\rceil S$	Premise (iii)
5.	$\rceil(P \vee Q)$	Modus tollens
6.	$\rceil P \wedge \rceil Q$	DeMorgan's law
7.	$\rceil Q$	Simplification RI_3

Thus the argument is valid.

4. *Test the validity of the following argument : If milk is black then every crow is white. If every crow is white, then it has four legs. If every crow has four legs, then every buffalo is white and brisk. The milk is black. Therefore the buffalo is white.*

Solution. : We name the propositions in the following way :

P denotes "The milk is black"

Q denotes "Every crow is white"

R denotes "Every crow has four legs"

S denotes "Every buffalo is white"

T denotes "Every buffalo is brisk"

The given premises are :

(i) $p \Rightarrow q$ (ii) $q \Rightarrow r$ (iii) $r \Rightarrow s \wedge t$ (iv) p

The conclusion is S.

Therefore,

1.	p	Premise (iv)
2.	$p \Rightarrow q$	Premise (i)
3.	q	Modus ponens RI_4
4.	$q \Rightarrow r$	Premise (ii)
5.	r	Modus ponens RI_4
6.	$r \Rightarrow s \wedge t$	Premise (iii)
7.	$s \wedge t$	Modus ponens RI_4
8.	s	Simplification RI_3

Hence, the argument is valid.

MISCELLANEOUS ILLUSTRATIVE EXAMPLES

1. *Prove that each of the following is a tautology :*

(a) $p \Rightarrow p$

(b) $p \wedge (p \Rightarrow q) \Rightarrow q$

(c) $\rceil p \Rightarrow (p \Rightarrow q)$

(d) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

(e) $[(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)] \Rightarrow [(p \vee q) \wedge (q \vee r) \wedge (r \vee p)]$

(f) $(p \Rightarrow q) \Rightarrow [(p \vee (q \wedge r) \Leftrightarrow [\{q \wedge (p \vee r)\}]$

Solution. (a) Truth table for $p \Rightarrow p$

p	p	$p \Rightarrow p$
T	T	T
F	F	T

Since the last column contains all the truth values T. Hence $p \Rightarrow p$ is a tautology.

(b) Truth table for $p \wedge (p \Rightarrow q) \Rightarrow q$

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$p \wedge (p \Rightarrow q) \Rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the last column contains all the truth values T. Hence $p \wedge (p \Rightarrow q) \Rightarrow q$ is a tautology.

(c) Truth table for $\urcorner p \Rightarrow (p \Rightarrow q)$

p	q	$\urcorner p$	$p \Rightarrow q$	$\urcorner p \Rightarrow (p \Rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Since the last column of the above table contains all the truth values T (True). Hence given statement is a tautology.

(d) Truth table for $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the last column contains all the truth values T (*i.e.*, True). Hence the given statement is a tautology.

(e) Truth table for $[(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)] \Rightarrow [(p \vee q) \wedge (q \vee r) \wedge (r \vee p)]$

p	q	r	$p \wedge q$	$q \wedge r$	$r \wedge p$	$p \vee q$	$q \vee r$	$r \vee p$	$[(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)]$	$[(p \vee q) \wedge (q \vee r) \wedge (r \vee p)]$	(e)
T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T	T	T	T
F	T	T	F	T	F	T	T	T	T	T	T
T	F	F	F	F	F	T	F	T	F	F	T
F	T	F	F	F	F	T	T	F	F	F	T
F	F	T	F	F	F	F	T	T	F	F	T
F	F	F	F	F	F	F	F	F	F	F	T

Since the last column of above, contains all the truth value T (*i. e.* true). Hence the given statement is a tautology.

(f) Truth table for $(p \Rightarrow q) \Rightarrow [(p \vee (q \wedge r) \Leftrightarrow \{q \wedge (p \vee r)\}]$

p	q	r	$q \wedge r$	$p \vee r$	$p \vee (q \wedge r)$	$q \wedge (p \vee r)$	$p \Rightarrow q$	$\{p \vee (q \wedge r) \Leftrightarrow \{q \wedge (p \vee r)\}$	(f)
T	T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	F	T	T	F	F	F	T
F	T	T	T	T	T	T	T	T	T
T	F	F	F	T	T	F	F	F	T
F	T	F	F	F	F	F	T	T	T
F	F	T	F	T	F	F	T	T	T
F	F	F	F	F	F	F	T	T	T

Since the last column of above table contains all the truth values T (*i. e.*, true). Hence the given statement is a tautology.

2. *The necessary and sufficient condition for two statements p and q to be logically equivalent is that $p \Leftrightarrow q$ is a tautology.*

Solution. If the truth value of $p \Leftrightarrow q$ is always T, then $p \Leftrightarrow q$ is a tautology. By the necessary and sufficient condition for the truth value of $p \Leftrightarrow q$ to be T, is that, the truth values of p and q should always be same. Thus p and q are logically equivalent.

3. *Prove that each of the following statement is a tautology :*

(a) $(p \wedge q) \Rightarrow q$ (b) $(p \vee \neg p)$ (c) $p \vee q \Rightarrow q \vee p$ (d) $\neg(p \wedge \neg p)$

(e) $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ (f) $(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow q) \wedge (q \Rightarrow p))$

(g) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

Solution. (a) Truth table for $(p \wedge q) \Rightarrow q$

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since $(p \wedge q) \Rightarrow q$ has its all truth values 'T'. Hence it is a tautology.

(b) Truth table for $p \vee \urcorner p$

p	$\urcorner p$	$p \vee \urcorner p$
T	F	T
F	T	T

Since $p \vee \urcorner p$ has always truth value T. Hence it is a tautology.

(c) Truth table for $p \vee q \Rightarrow q \vee p$

p	q	$p \vee q$	$q \vee p$	$p \vee q \Rightarrow q \vee p$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

Since $p \vee q \Rightarrow q \vee p$ has its all truth values T. Hence it is a tautology.

(d) Truth table for $\urcorner(p \wedge \urcorner p)$

p	$\urcorner p$	$p \wedge \urcorner p$	$\urcorner(p \wedge \urcorner p)$
T	F	F	T
F	T	F	T

Since $\urcorner(p \wedge \urcorner p)$ has its all truth values T. Hence it is a tautology.

(e) Truth table for $(p \Rightarrow q) \Leftrightarrow (\urcorner p \vee q)$

p	q	$\urcorner p$	$p \Rightarrow q$	$\urcorner p \vee q$	$(p \Rightarrow q) \Leftrightarrow (\urcorner p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

The last column contains all the truth values 'T'. Hence it is a tautology.

(f) Truth table for $(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$(p \Leftrightarrow q) \Leftrightarrow [(p \Rightarrow q) \wedge (q \Rightarrow p)]$
T	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

The last column of above table contains all the truth values T. Hence, the given statement is a tautology.

(g)

p	q	$\rceil p$	$\rceil q$	$p \Rightarrow q$	$\rceil q \Rightarrow \rceil p$	$(p \Rightarrow q) \Leftrightarrow (\rceil q \Rightarrow \rceil p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

The last column contains all the truth values. T. Hence, it is a tautology.

4. *Check whether following are tautology or contradiction :*

(a) $[(p \wedge q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow r) \vee (q \Rightarrow r)]$ *(b)* $(p \Rightarrow q \wedge r) \Rightarrow (\rceil r \Rightarrow \rceil p)$

(c) $(p \Leftrightarrow q \wedge r) \Rightarrow (\rceil r \Leftrightarrow \rceil p)$ *(d)* $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (q \wedge r)$

(e) $([p \Leftrightarrow q) \vee (r \Rightarrow p)$ *(f)* $(p \vee q \vee r) \Leftrightarrow [(\rceil p \Rightarrow q) \Rightarrow q) \Rightarrow r]$

(g) $[(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \rceil q]$ *(h)* $[((p \Rightarrow q) \vee p) \wedge q] \Rightarrow q$

(i) $p \wedge (q \wedge \rceil p)$

Solution. (a) Truth table for $[(p \wedge q) \Rightarrow r] \Leftrightarrow [(p \Rightarrow r) \vee (q \Rightarrow r)]$

p	q	r	$p \wedge q$	$p \Rightarrow r$	$q \Rightarrow r$	$(p \wedge q) \Rightarrow r$	$(p \Rightarrow r) \vee (q \Rightarrow r)$	(a)
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	F
T	F	T	F	T	T	T	T	T
F	T	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

The last column of above table does not contain all the truth values is T. Hence, the given statement is not a tautology.

(b) Truth table for $(p \Rightarrow q \wedge r) \Rightarrow (\rceil r \Rightarrow \rceil p)$

p	q	r	$\rceil p$	$\rceil r$	$q \wedge r$	$p \Rightarrow q \wedge r$	$\rceil r \Rightarrow \rceil p$	$(p \Rightarrow q \wedge r) \Rightarrow (\rceil r \Rightarrow \rceil p)$
T	T	T	F	F	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	F	F	T	T
F	T	T	T	F	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	F	T	T	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	F	T	T	F	T	T	T

The given statement contains all its truth values T. Hence, it is a tautology.

(c) Truth table for $(p \Leftrightarrow q \wedge r) \Rightarrow (\neg r \Rightarrow \neg p)$

p	q	r	$\neg p$	$\neg r$	$q \wedge r$	$p \Leftrightarrow q \wedge r$	$\neg r \Rightarrow \neg p$	$(p \Leftrightarrow q \wedge r) \Rightarrow (\neg r \Rightarrow \neg p)$
T	T	T	F	F	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	F	F	T	T
F	T	T	T	F	T	F	T	T
T	F	F	F	T	F	F	F	T
F	T	F	T	T	F	T	T	T
F	F	T	T	F	F	T	T	F
F	F	F	T	T	F	T	T	T

Similarly, the given statement does not contains always truth value T. Hence, it is not a tautology.

(d) Truth table for $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

p	q	r	$p \wedge q$	$q \vee r$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \wedge (q \vee r)$ $\Rightarrow (p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T	T
T	F	T	F	T	T	T	T	T
F	T	T	F	T	F	F	F	T
T	F	F	F	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	F	T	F	F	F	T
F	F	F	F	F	F	F	F	T

The last column contains always truth values T. Hence, the given statement is a tautology.

(e) Truth table for $(p \Rightarrow q) \vee (r \Rightarrow p)$

p	q	r	$p \Rightarrow q$	$r \Rightarrow p$	$(p \Rightarrow q) \vee (r \Rightarrow p)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	F	T	T
F	T	T	T	F	T
T	F	F	F	T	T
F	T	F	T	T	T
F	F	T	T	F	T
F	F	F	T	T	T

Last column of above table contains all its truth values T. Hence, given statement is a tautology.

(f) Truth table for $(p \vee q \vee r) \Leftrightarrow [\{(\neg p \Rightarrow q) \Rightarrow q\} \Rightarrow r]$

p	q	r	$\neg p \Rightarrow q$	$(\neg p \Rightarrow q) \Rightarrow q$	$(((\neg p \Rightarrow q) \Rightarrow q) \Rightarrow r)$	$p \vee q \vee r$	$(((\neg p \Rightarrow q) \Rightarrow q) \Rightarrow r) \to r$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F
T	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	F
F	F	T	T	F	T	T	T
F	F	F	T	F	T	F	F

In the above table, the last column contains all the truth values T. Hence, the given statement is not a tautology.

(g) Truth table for $[(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \neg q]$

p	q	$\neg q$	$p \wedge q$	$(p \wedge q) \Rightarrow p$	$q \wedge \neg q$	$[(p \wedge q) \Rightarrow p] \Rightarrow [q \wedge \neg q]$
T	T	F	T	T	F	F
T	F	T	F	T	F	F
F	T	F	F	T	F	F
F	F	T	F	T	F	F

The last column contains all the truth values F. Hence, the given statement is contradiction.

(h) Truth table for $[\{(p \Rightarrow q) \vee p\} \wedge q] \Rightarrow q$

p	q	$p \Rightarrow q$	$(p \Rightarrow q) \wedge p$	$[\{(p \Rightarrow q) \vee p\} \wedge q]$	$[\{(p \Rightarrow q) \vee p\} \wedge q] \Rightarrow q$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	T	F	T

In above table, last column contains all the truth values T. Hence, given statement is a tautology.

(i) Truth table for $p \wedge (q \wedge \neg p)$

p	q	$\neg p$	$q \wedge \neg p$	$p \wedge (q \wedge \neg p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

The last column contains all the truth value F. Hence, given statement is a contradiction.

5. *Determine whether the following formulas are tautology, contradiction satisfiable :*

(a) $(p \wedge (p \Rightarrow q)) \Rightarrow \neg q$ *(b)* $((p \Rightarrow q) \wedge (q \Rightarrow r)) \wedge (p \wedge \neg r)$

(c) $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

Solution. (a) $(p \wedge (p \Rightarrow q)) \Rightarrow \neg q$

p	q	$\neg q$	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow \neg q$
T	T	F	T	T	T
T	F	T	F	F	T
F	T	F	T	F	F
F	F	T	T	F	T

(b) $((p \Rightarrow q) \wedge (q \Rightarrow r)) \wedge (p \wedge \neg r)$

p	q	r	$\neg r$	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$ (1)	$p \wedge \neg r$ (2)	(1) $\wedge$ (2)
T	T	T	F	T	T	T	F	F
T	T	F	T	T	F	F	T	F
T	F	T	F	F	T	F	F	F
F	T	T	F	T	T	T	F	F
T	F	F	T	F	T	F	T	F
F	T	F	T	T	F	F	F	F
F	F	T	F	T	T	T	F	F
F	F	F	T	T	T	T	F	F

(c) $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \wedge \neg q$	$q \vee (p \wedge \neg q)$	$\neg p \wedge \neg q$	$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

6. *Prove that* $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$.

Solution. Construct the truth table as follows :

p	q	r	$p \Rightarrow q$	$p \vee r$	$q \vee r$	$(p \Rightarrow q) \vee r$	$(p \vee r) \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T

T	F	F	F	T	F	F	F
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	F	F	T	T

The last two columns have same truth values. Thus corresponding statements of the columns are logically equivalent and

$$(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r).$$

7. *Prove that* $(p \Rightarrow q) \vee (p \Rightarrow r) \equiv p \Rightarrow (q \vee r)$.

Solution. Construct the truth table as follows :

p	q	r	$q \vee r$	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$	$p \Rightarrow (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

The last two columns have same truth values. Hence,

$$(p \Rightarrow q) \vee (p \Rightarrow r) \text{ and } p \Rightarrow (q \vee r)$$

are logically equivalent and $(p \Rightarrow q) \vee (p \Rightarrow r) \equiv p \Rightarrow (q \vee r)$.

8. *Disprove that* $(p \vee q) \vee (p \wedge q) \equiv p$

Solution. Construct the truth table

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \vee (p \wedge q)$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

The first and the last columns corresponding to the given statement do not contain same truth values. Hence, $(p \vee q) \vee (p \wedge q)$ and p are not logically equivalent.

9. *Prove that :*

(a) $p \Rightarrow q \equiv \neg p \vee q$

(b) $p \Rightarrow (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$

(c) $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

(d) $\neg(p \Leftrightarrow q) \equiv \neg p \Leftrightarrow q \equiv p \Leftrightarrow \neg q$.

Solution. (a) Construct the truth table

p	q	$\rceil p$	$p \Rightarrow q$	$\rceil p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last two columns corresponding to the given statements have same truth values. Hence, $(p \Rightarrow q)$ and $\rceil p \vee q$ are logically equivalent and $p \Rightarrow q \equiv \rceil p \vee q$.

(b) Construct the truth table

p	q	r	$p \wedge q$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$(p \wedge q) \Rightarrow r$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
T	F	F	F	T	T	T
F	T	F	F	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

The last two columns corresponding to the given statements have same truth values.

$\therefore$ $p \Rightarrow (q \Rightarrow r)$ and $(p \wedge q) \Rightarrow r$ are logically equivalent and $p \Rightarrow (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$.

(c) Construct the truth table

p	q	r	$q \wedge r$	$p \Rightarrow q$	$p \Rightarrow r$	$p \Rightarrow (q \wedge r)$	$(p \Rightarrow q) \wedge (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
F	T	T	T	T	T	T	T

The last two columns have same truth values. Hence, the statements $p \Rightarrow (q \wedge r)$ and $(p \Rightarrow q) \wedge (p \Rightarrow r)$ are logically equivalent and

$$p = (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r).$$

(d) Construct the truth table

p	q	$\rceil p$	$\rceil q$	$p \Leftrightarrow q$	$\rceil(p \Leftrightarrow q)$	$\rceil p \Leftrightarrow q$	$p \Leftrightarrow \rceil q$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	T	F	F	F

The last three columns have same truth values. Hence, $\rceil(p \Leftrightarrow q)$, $(\rceil p \Leftrightarrow q)$ and $p \Leftrightarrow \rceil q$ are logically equivalent and $\rceil(p \Leftrightarrow q) \equiv \rceil p \Leftrightarrow q \equiv p \Leftrightarrow \rceil q$.

10. *Establish the equivalence* $(p \Rightarrow q) \Rightarrow (p \wedge q) = (\neg p \Rightarrow q) \wedge (q \Rightarrow p)$.

Solution. Construct the truth table

p	q	$\neg p$	$p \Rightarrow q$	$p \wedge q$	$\neg p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \Rightarrow (p \wedge q)$	$(\neg p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	F	T	F	F	F
F	F	T	T	F	F	F	F	F

The last two columns have same truth values. Hence, $(p \Rightarrow q) \Rightarrow (p \wedge q)$ and $(\neg p \Rightarrow q) \wedge (q \Rightarrow p)$ are logically equivalent and

$$(p \Rightarrow q) \Rightarrow (p \wedge q) = (\neg p \Rightarrow q) \wedge (q \Rightarrow p).$$

11. *Establish the equivalence and write its dual equivalence :*

(a) $(p \wedge q) \vee [\neg p \vee (\neg p \vee q)] \equiv \neg p \vee q$ *(b)* $\neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Solution. (a) LHS $= (p \wedge q) \vee [\neg p \vee (\neg p \vee q)]$

$$\equiv (p \wedge q) \vee [(\neg p \vee \neg p) \vee q] = (p \wedge q) \vee [(\neg p \vee q)]$$
$$\equiv [p \vee (\neg p \vee q)] \wedge [q \vee (\neg p \vee q)]$$
$$\equiv [p \vee (\neg p \vee q)] \wedge (\neg p \vee q)$$
$$\equiv \neg p \vee q \quad \text{(by absorption)} = \text{RHS}$$

Hence, $(p \wedge q) \vee \neg p\, \neg p \vee (p \vee q)] \equiv \neg p \vee q$.

The dual of this statement is $(p \vee q) \wedge [\neg p \wedge (p \wedge q)] \equiv \neg p \wedge q$

(b) LHS $= \neg p \wedge (\neg q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

$$\equiv [(\neg p \wedge \neg q) \wedge r] \vee [\{(q \wedge r) \vee p) \wedge \{(q \wedge r) \vee r\}]$$
$$\equiv [(\neg p \wedge \neg q) \wedge r] \vee [\{(q \wedge r) \vee p) \wedge r] \qquad \text{(By absorption)}$$
$$\equiv [(\neg p \wedge \neg q) \wedge r] \vee [\{(q \wedge r) \wedge r\} \vee (p \wedge r)]$$
$$\equiv [(\neg p \wedge \neg q) \wedge r] \vee [(q \wedge r) \vee (p \wedge r)]$$
$$\equiv [(\neg p \wedge \neg q) \vee (q \vee p)] \wedge r$$
$$\equiv [(\neg p \vee \neg q) \vee (q \vee p)] \wedge r$$
$$\equiv r$$

Now to get dual of the given statement, interchange $\wedge$ and $\vee$, we get

$$\neg p \wedge (\neg q \vee r) \wedge r)\, q \wedge r \wedge (p \vee r) \equiv r.$$

12. *Show that the following argument is valid*

$$\begin{array}{l} p \\ p \Rightarrow q \\ \underline{q \Rightarrow r} \\ r \end{array}$$

Solution. First Method : Let $f \equiv [p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow r$.

Now, construct the truth table for f

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)$	(f)
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
F	T	T	T	T	F	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

The last column contains all the truth values T. Hence, f is a tautology and hence the given argument is valid.

Second Method :

Let $f \equiv [p \wedge (p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow r$

$\equiv [p \wedge (\neg p \vee q) \wedge (\neg q \vee r)] \Rightarrow r \equiv \neg[p \wedge (\neg p \vee q) \wedge (\neg q \vee r)] \vee r$

$\equiv [\neg p \vee \neg(\neg p \vee q) \vee \neg(\neg q \vee)] \vee r \equiv [\neg p \vee (p \wedge \neg q) \vee (q \wedge \neg r)] \vee r$

$\equiv \neg p \vee \neg q \vee \neg r \vee r$

Thus, $f \equiv \neg p \vee \neg q \vee \neg r \vee r = 1$

Since f reduced to 1, this also shows that the argument is valid.

Third Method : Consider the following sequences of arguments

p a premise
$\underline{p \Rightarrow q}$ a premise
q by the law of detachment
$\underline{q \Rightarrow r}$ a premise
r by the law of detachment

Thus are argument is valid.

13. *Check the validity of the argument*

$$\begin{array}{c} p \Rightarrow q \\ \underline{r \Rightarrow \neg q} \\ p \Rightarrow \neg r \end{array}$$

Solution. Let $f \equiv [(p \Rightarrow q) \wedge (r \Rightarrow \neg q)] \Rightarrow (p \Rightarrow \neg r)$

Now, we construct truth table for f

p	q	r	$\neg q$	$\neg r$	$p \Rightarrow q$	$r \Rightarrow \neg q$	$(p \Rightarrow q) \wedge (r \Rightarrow \neg q)$	$(p \Rightarrow \neg r)$	(f)
T	T	T	F	F	T	F	F	F	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	F	T	F	F	T
F	T	T	F	F	T	F	F	T	T
T	F	F	T	T	F	T	F	T	T
F	T	F	F	T	T	T	T	T	T
F	F	T	T	F	T	T	T	T	T
F	F	F	T	T	T	T	T	T	T

Since f column contains only T, the argument is valid.

14. *Test the validity of the argument :*

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

∴ The opposite angles are not equal.

Solution. The statement above the horizontal line are two premises. The statement below the horizontal line is the conclusion.

Let p: Two sides of a triangle are equal.

q: The opposite angles of a triangle are equal.

The given argument in symbolic form can be written as

$$\begin{array}{c} p \Rightarrow q \text{ (a premise)} \\ \underline{\neg p \text{ (a premise)}} \\ \neg q \text{ (conclusion)} \end{array}$$

We shall construct the truth table for statement.

$$[(p \to q) \wedge \neg p] \to \neg q$$

p	q	$p \Rightarrow q$	$\neg p$	$\neg q$	$(p \Rightarrow q) \wedge \neg p$	$[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

The last column in the table shows that $[(p \Rightarrow q) \wedge \neg p] \Rightarrow \neg q$ is not a tautology. Hence the given argument is not valid.

15. *State whether the argument given below is valid or not valid. If it is valid, identify the tautology used :*

I will become famous or I will be the writer.
I will not be a writer.
∴ I will become famous.

Solution. Let p : I will become famous.

q : I will be a writer.

∴ The given argument in symbolic form can be written as

$p \vee q$	(a premise)
$\sim q$	(a premise)
p	(conclusion)

The given argument, *i.e.* $(p \vee q) \wedge (\sim q) \to p$ will be valid if the statement $[(p \vee q) \wedge (\sim q)] \to p$ is a tautology. Now we construct the truth table for the above statement

p	q	$p \vee q$	$\sim q$	$(p \vee q) \wedge (\sim q)$	$[(p \vee q) \wedge (\sim q)] \to p$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	F	T	F	T

Since last column contains only Trues, hence the given argument is valid.

Exercise 2.2

1. Which of the following are statements :
   (a) Grass is yellow
   (b) Is the number 5 a prime ?
   (c) Give me a book
   (d) $\{x : x^2 = 9\} = \{3, -3\}$
   (e) $x^2 + y^2 \geq 0$
   (f) White roses are beautiful.
   (g) If dogs can bark, then no home guarded by a dog needs to fear intruders.
   (h) $\{7x + 5y = 12 : (x, y) \equiv (1, 1), x, y \in N\}$
2. Let p be the proposition (statement) "x is an even number" and let q be the statement "x is the product of two integers". Translate into symbols each of the following statements
   (a) Either x is an even number, or x is a product of two integers.
   (b) x is an odd number, and x is a product of two integers.

(c) Either x is an even number and a product of integers, or x is an odd number and is not a product of integers.

(d) x is neither an even number nor a product of integers.

3. Write in reasonable English, the negation of each of the following statements :

(a) Ice is cold, and I am tired.

(b) Either good health is desirable, or I have been misinformed.

(c) Oranges are not suitable for use in vegetable salads.

(d) There is a number which, when added to 6, gives a sum of 13.

4. Define compound statement and explain it by an example.

5. If $p \equiv$ It is about to 6 : 45 p.m.; $q \equiv$ Sangam train is about to depart.

(a) $p \wedge q$ (b) $p \vee q$

(c) $q \vee \neg p$ (d) $p \wedge \neg q$

(e) $\neg(p \wedge q)$ (f) $\neg(p \vee q)$

(g) $\neg p \Rightarrow q$ (h) $\neg q \Rightarrow p$

6. Write the negation of the following :

(a) If the determinant of a system of linear equation is zero, then, either the system has no solution or has an infinite number of solution.

(b) Either today is not a Friday or today is not a Sunday.

7. Use truth tables to verify

(a) The associative law for disjunction

(b) The associative law for conjunction

(c) The distributive law for $\wedge$ and $\vee$, and

(d) The two laws of absorption.

8. Construct the truth tables for the following statements :

(a) $\neg[\neg r \vee (p \wedge q)]$

(b) $p \vee (q \wedge r)$

(c) $(p \wedge \neg q) \vee (\neg p \wedge q)$

(d) $p \vee q \vee \neg p$

(e) $(p \wedge q) \vee (\neg p \wedge \neg q)$

(f) $\neg p = \neg q$

(g) $(p \vee q) \wedge \neg(p \wedge q)$

9. Prove De Morgan's laws using truth tables.

10. Construct the truth table for the contrapositive of $p \Rightarrow q$.

11. Designate suitable simple statements p and q and translate the following statement into symbols :

(a) If lemons are expensive and sugar is cheap, then sour lemonade is rarely seen.

(b) Sour lemonade is often seen unless sugar is cheap.

(c) A necessary condition for lemons to be cheap is that sugar is expensive.

(d) Sour lemonade is rarely seen only if sugar is cheap.

12. Determine the validity of the following argument : "If wages increase then there will be inflation. The cost of living will not increase, if there is no inflation. Wages will increase, therefore the cost of living will increase.

13. Without using truth table, show that

$$((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$$

is a tautology.

14. Determine the validity of the following argument : "If Mary runs for office, she will be elected. If Mary attends the meeting, she will run for office. If mary attends the meeting, she will run for office. Either Mary will attend the meeting or she will go to India, but Mary cannot go to India. Thus Mary will be elected.

15. Prove that the following are tautologies :

(a) $(p \Leftrightarrow q) \wedge (q \Leftrightarrow r) \Rightarrow (p \Leftrightarrow r)$

(b) $\neg(p \wedge \neg p)$

(c) $(p \Rightarrow q) \vee r \Leftrightarrow (p \vee r) \Rightarrow (q \vee r)$

(d) $(p \Rightarrow q) \wedge \neg q$

(e) $[(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(f) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

16. Prove the following equivalences and write its dual.

(a) $\neg(p \Leftrightarrow q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

(b) $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$

(c) $p \wedge q \equiv q \wedge p$

(d) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(e) $(p \wedge \neg q \vee q) \equiv p \vee q$

(f) $(p \wedge q \Rightarrow r) \equiv (p \Rightarrow r) \vee (q \Rightarrow r)$

17. Define the tautology and duality with example.

18. Prove that conditional operation is neither commutative nor associative.

19. Prove that the following formula is not a tautology.

$$[(p \wedge r) \vee (q \wedge \neg r)] \Leftrightarrow [(p \wedge r) \vee (\neg q \wedge \neg r)].$$

20. Simplify the following :

(a) $(p \vee q) \Leftrightarrow \neg p$

(b) $p \vee (p \wedge q)$

(c) $\neg(p \vee q) \vee (\neg p \wedge q)$

(d) $\neg(\neg p \wedge q) \wedge (\neg p \vee q) \wedge (p \vee q)$.

21. Check the validity of the following :

(a) $p \Rightarrow q$
$r \Rightarrow \rceil q$
$\overline{}$
$p \Rightarrow \rceil r$

(b) p
q
$\rceil p \Rightarrow r$
$q \Rightarrow \rceil r$
$\overline{}$
$\rceil r$

(c) $q \Rightarrow p$
$q \vee s$
$\rceil s$
$\overline{}$
p

(d) p
$\rceil p \vee \rceil s) \Rightarrow (\rceil p \wedge \rceil r)$
$\overline{}$
s

(e) $r \Rightarrow \rceil q$
$p \Rightarrow q$
$\rceil r \Rightarrow s$
$\overline{}$
$p \Rightarrow s$

(d) p
$\rceil q \vee r$
$\rceil p \Rightarrow q$
$\overline{}$
r

22. Write the dual of the following expression
$(x \wedge 1) \wedge (0 \vee x')$.

ANSWERS

1. (a) Yes, (b) Yes, (c) Yes, (d) Yes, (e) Yes, (f) No, (g) Yes, (h) Yes,

2. (a) $p \vee q$, (b) $\rceil p \wedge q$, (c) $(p \wedge q) \vee (\rceil p \wedge \rceil q)$, (d) $\rceil p \wedge \rceil q$

3. (a) Either ice is not cold or I am not tired, (b) Good health is not desirable and I have not been informed, (c) Oranges are suitable for use in vegetable salads, (d) There is no number which when did not add to 6, does not give a sum of 13

5. (a) It is about 6 : 45 p.m. and Sangam train is about to depart, (b) Either it is about 6 : 45 p.m. or Sangam train is about to depart, (c) Either Sangam Train is about to depart or it is not about to 6 : 45 p.m., (d) It is about to 6 : 45 p.m. and Sangam train is not about to depart, (e) It is false that it is about to 6 : 45 p.m. and Sangam train is about to depart, (f) Neither it is about to 6 : 45 p.m. nor Sangam train is about to depart, (g) If it is not about to 6 : 45 p.m., then Sangam train is about to depart, (h) If Sangam train is not about to depart, then it is about 6 : 45 p.m.

6. (a) It is false that if the determinant of a system of linear equation is zero, then either the system has no solution or has an infinite number of solutions, (b) Today is a Friday and today is a Sunday.

8. (a)

p	q	r	$\rceil r$	$p \wedge q$	$\rceil r \vee (p \wedge q)$	$\rceil[\rceil r \vee (p \wedge q)]$
T	T	T	F	T	T	F
T	T	F	T	T	T	F
T	F	T	F	F	F	T
F	T	T	F	F	F	T
T	F	F	T	F	T	F
F	T	F	T	F	T	F
F	F	T	F	F	F	T
F	F	F	T	F	T	F

(b)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	F	T
T	T	F	T	T
T	F	T	F	F
F	T	T	F	F

T	F	F	T	F
F	T	F	T	F
F	F	T	F	F
F	F	F	T	F

(c)

p	q	r	$\rceil p$	$\rceil q$	$p\wedge \rceil q$	$\rceil p\wedge q$	$(p\wedge \rceil q)\vee(\rceil p\wedge q)$
T	T	T	F	F	F	F	F
T	T	F	F	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	T	F	F	T	T
T	F	F	F	T	T	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	F	F	F
F	F	F	T	T	F	F	F

(d)

p	q	$\rceil p$	$p\vee q\vee \rceil p$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	T

(e)

p	q	$\rceil p$	$\rceil q$	$p\wedge q$	$\rceil p\wedge \rceil q$	$(p\wedge q)\vee(\rceil p\wedge \rceil q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

(f)

p	q	$\rceil p$	$\rceil q$	$\rceil p\Leftrightarrow \rceil q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(g)

p	q	$p \vee q$	$p \wedge q$	$\neg p \wedge q$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

11. If p = lemons are expensive, q = sugar is cheap, r = Lemonade is rarely seen

(a) $(p \wedge q) \Rightarrow r$, (b) $\neg q \Rightarrow \neg r$, (c) $\neg p \Rightarrow \neg q$, (d) $r \Rightarrow q$

16.(a) $\neg(p \Leftrightarrow q) \equiv (p \vee \neg q) \wedge (\neg p \vee q)$, (b) $p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$,

(c) $p \vee q \equiv q \vee p$, (d) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(e) $(p \vee \neg q) \wedge q \equiv p \wedge q$, (f) $(p \vee q \Rightarrow r) \equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$

20.(a) $\neg p \wedge q$, (b) p, (c) $\neg p$, (d) $p \wedge q$

21.(a) valid, (b) valid, (c) valid, (d) valid, (e) valid, (f) invalid.

22. $(x \vee 1) \vee (0 \wedge x')$