Discrete Mathematics
Code - PCC - CS401
Stream - CSE/IT

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Module – V
Graph Theory contd…

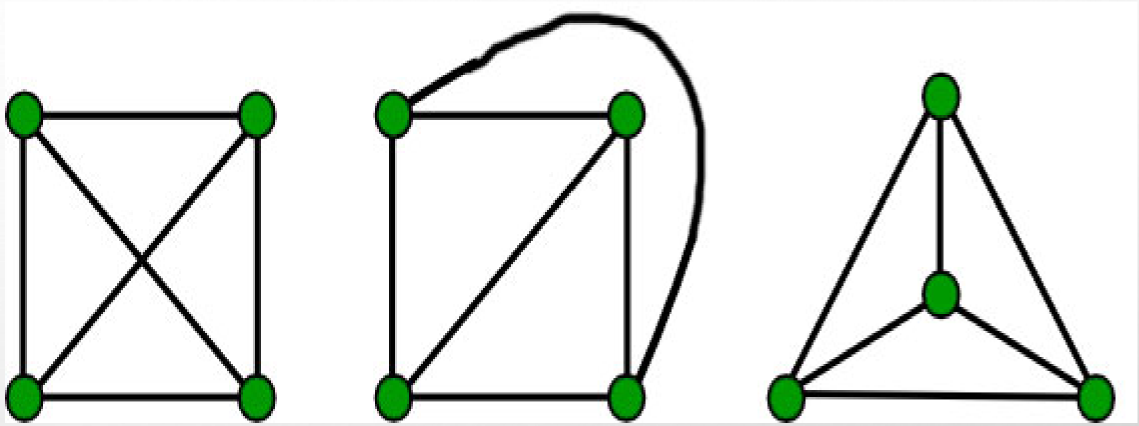
- Planar graphs
- Euler's formula (n - e + r = 2) for connected planar graph and its generalization for graphs with connected components,
- Kuratowski’s graphs

- Graph coloring
- Chromatic numbers of C_n , K_n , $K_{m,n}$
- Upper bounds of chromatic numbers (statements only), Brook’s theorem, Vizing’s theorem,
- Four color theorem
- Clique number & Perfect graph
- Chromatic index

Planar Graph

- Plane graph (or embedded graph) - A graph that can be drawn on a plane without edge crossing, is called a Plane graph

- Planar graph - A graph is called Planar, if it is isomorphic with a Plane graph.

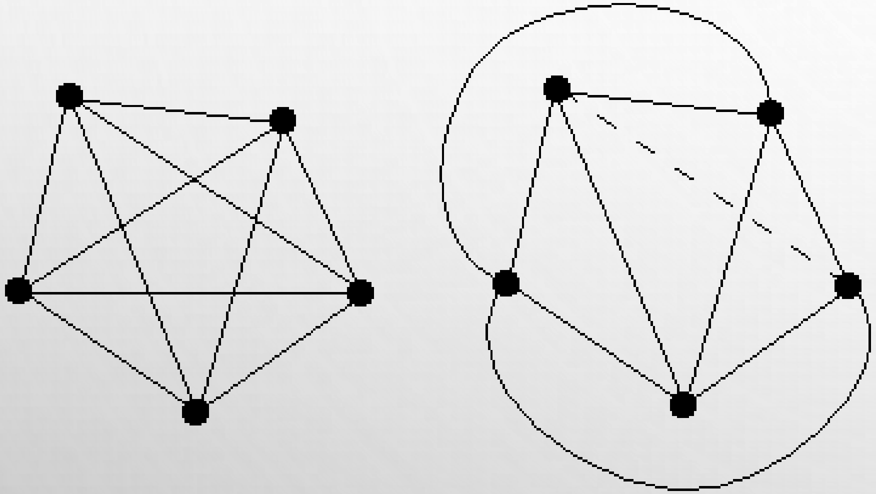


G_1 **G_2** **G_3**

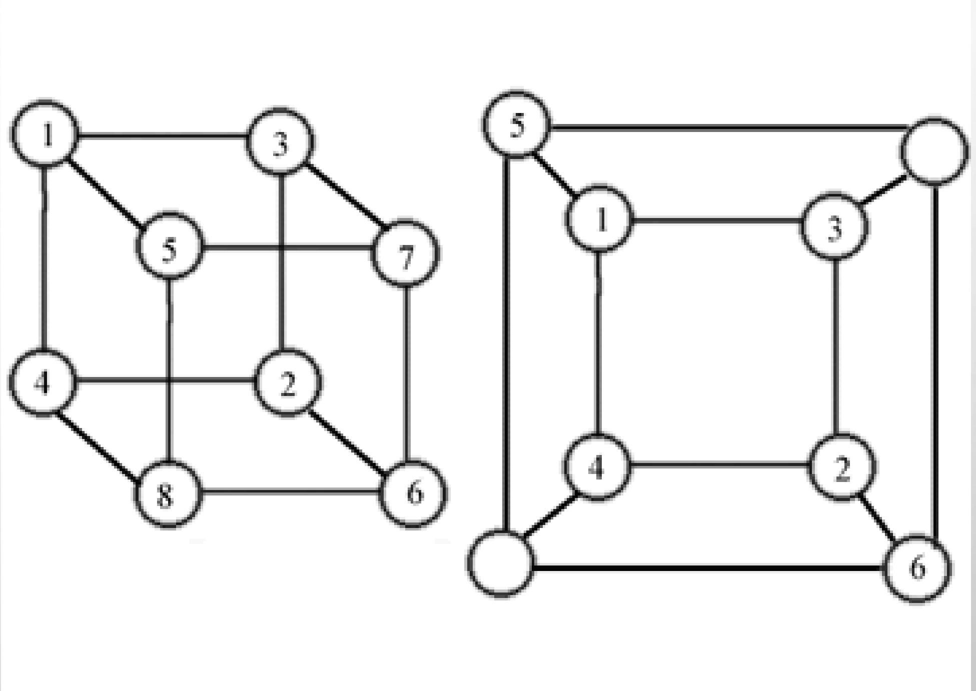
❑ Example –

- G_2 and G_3 are plane graphs, but G_1 is not
- G_1 is planar, as it is isomorphic to G_2 and G_3
- G_2 and G_3 are called planar embeddings of K_4

Planar & Non – Planar Representation of a Graph



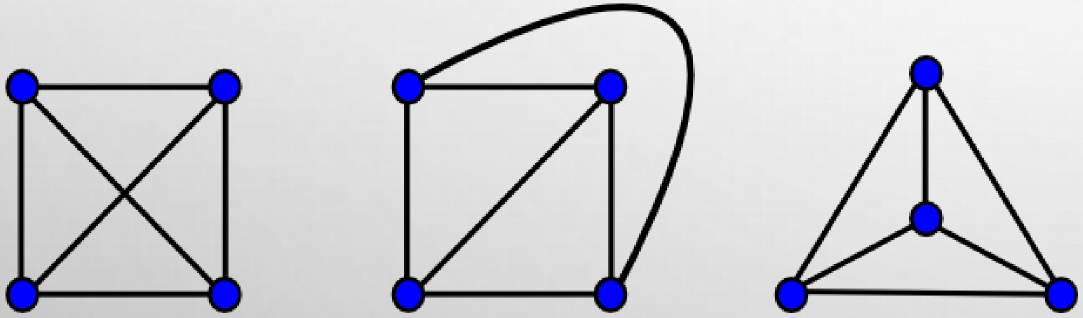
Non – Planar Graph



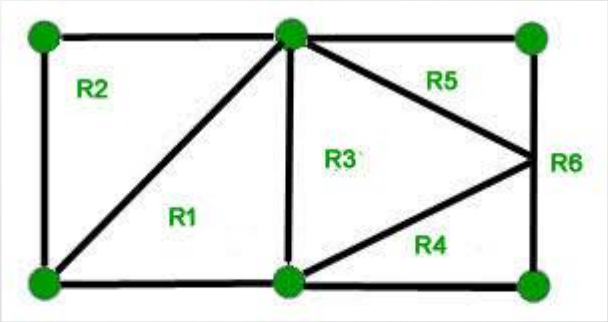
Planar Graph

- Faces or Regions –

A planar representation of a graph divides the plane into a number of connected regions, called faces. These regions are bounded by the edges except for one regionthat is unbounded.



4 regions



6 Regions

Euler's Theorem - If G is a simple connected planar graph, then the n - e + f = 2, where n, e, f are total number of vertices, edges and faces of G respectively.

Proof - by induction on f :

- For f = 1, G is a tree as a tree has no cycle so only one unbounded region exists for it. For every tree, e = n - 1, so n – e = 1. Hence n - e + f = 1 + 1 = 2 and the formula holds.
- Suppose it holds for all planar graphs with less than f faces and suppose that g has f ≥ 2 faces.
- Let (u,v) be an edge of G which lies on a cycle. Such an edge must exist because G has more than one face. (u,v) lies on the boundary of exactly two faces f_1, f_2 of G.

- Let G' = G - (u, v). Then the removal of (u, v) will cause the two faces f_1, f_2 separated by (u, v) to combine, forming a single face.
- Hence G' = G - (u, v) is a planar embedding of a simple connected graph with one less face than g. Therefore,
- f(G') = f(G) – 1, n(G') = n(G), e(G') = e(G) – 1.
- But by the induction hypothesis, n(G') - e(G') + f(G') = 2 and so, by substitution we get n(G) - e(G) + f(G) = 2.

Hence, by induction, Euler's formula holds for all simple connected planar graphs.

Lemma 1 - For any simple connected planar graph G, $\sum_i d(f_i) = 2\,e$.

Proof. Each edge contributes 1 to each face it is a bound, so it contributes 2 to the total sum. So the e(G) edges contribute 2e(G) to the total sum.

Lemma 2 - For any simple connected planar graph G, with e ≥ 3, the following holds: e ≤ 3n - 6.

Proof - Each face of any planar graph G is bounded by at least three edges, hence $\sum_i d(f_i) \geq 3\,f$

- Also from lemma 1, $\sum_i d(f_i) = 2\,e$. Therefore 2e ≥ 3f which implies f ≤ $\frac{2}{3}$ e.
- Using Euler's formula n - e + f = 2 which implies f = 2 - n + e
- So, 2 - n + e ≤ $\frac{2}{3}$ e. Hence $\frac{1}{3}$ e ≤ n – 2 and it follows that e ≤ 3n - 6

Lemma 3 - For any simple connected bipartite graph G, with e ≥ 3, the following holds   e ≤ 2n - 4.

Proof. As G is bipartite, so each face of every embedding of G has at least 4 edges (every bipartite graph has cycle of only even length), hence $\sum_i d(f_i) \geq 4\ f$

- Also from lemma 1, $\sum_i d(f_i) = 2$ e. Therefore 2e ≥ 4f which implies f ≤ $\frac{1}{2}$ e.
- Using Euler's formula n - e + f = 2 which implies f = 2 - n + e
- So, 2 - n + e ≤ $\frac{1}{2}$ e. Hence $\frac{1}{2}$ e ≤ n – 2 and it follows that e ≤ 2n - 4

Complete Graphs : Planar or Non – Planar ??

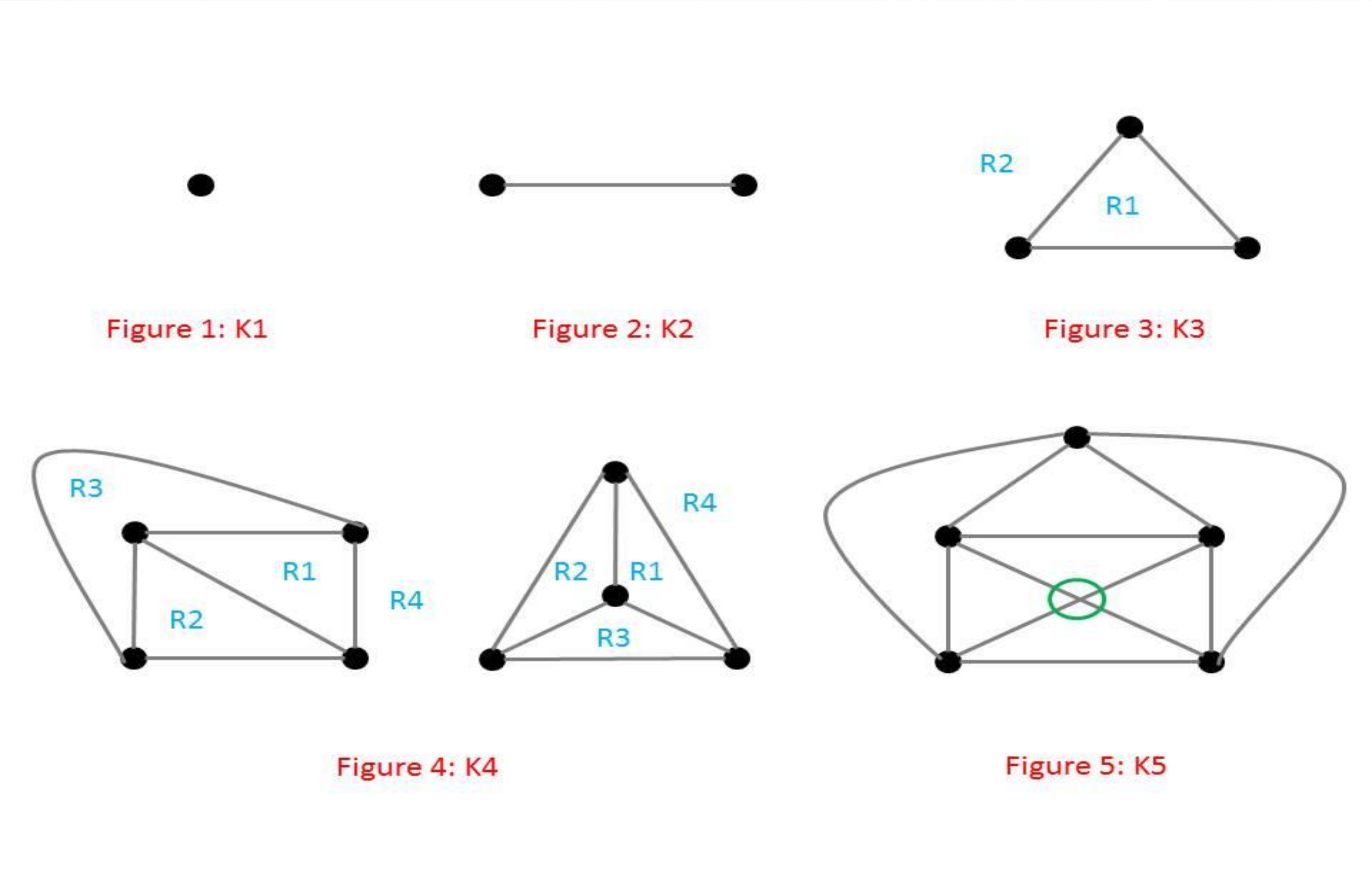


Figure 1: K1
Figure 2: K2
Figure 3: K3
Figure 4: K4
Figure 5: K5

- For K_5, we have

  $n = 5$

  $e = \frac{n(n-1)}{2} = \frac{5.4}{2} = 10$

  $f = ??$

- Applying Lemma-2

  $e = 10 \geq 3n - 6 = 3.5 - 6 = 9$

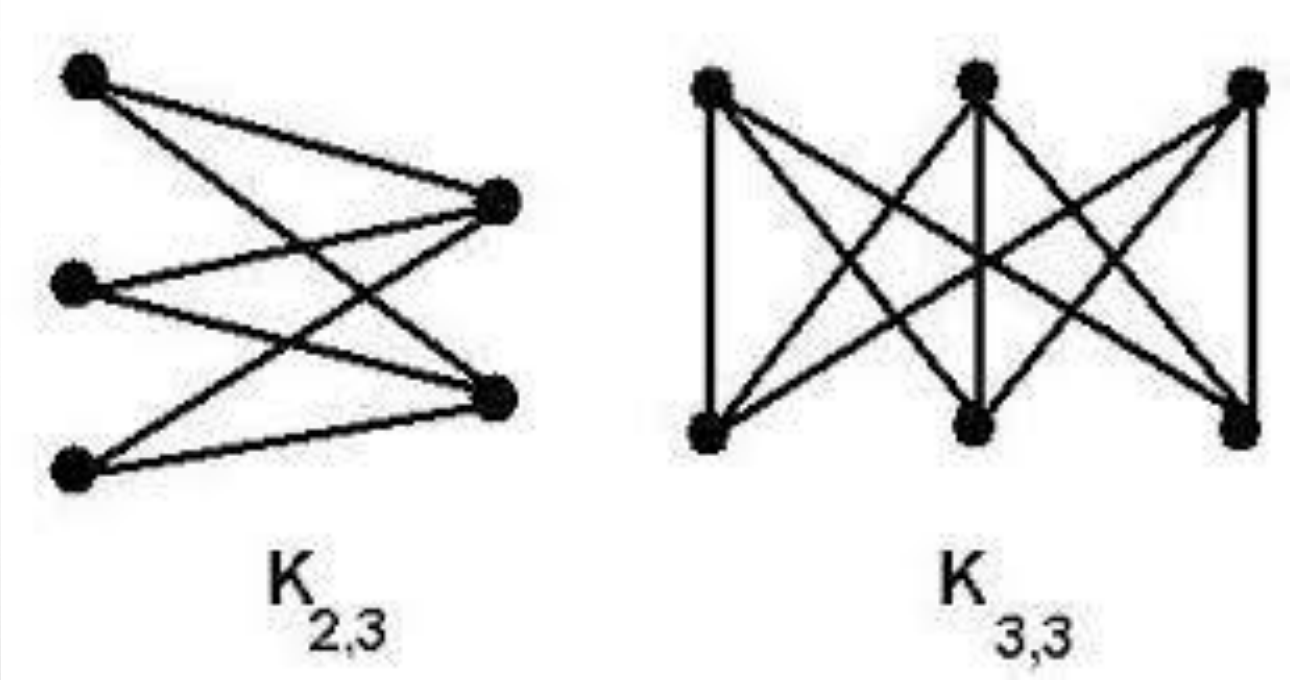
  Doesn't satisfy. Hence not Planar

- <u>What about any complete graph K_n with $n \geq 5$?</u>

Bipartite Graphs : Planar or Non – Planar ??

- For $K_{2,3}$ we have
  $e = 6$ & $n = 5$

But $e = 6 = 2.5 - 4 = 2n - 4$
Lemma – 2 satisfies.

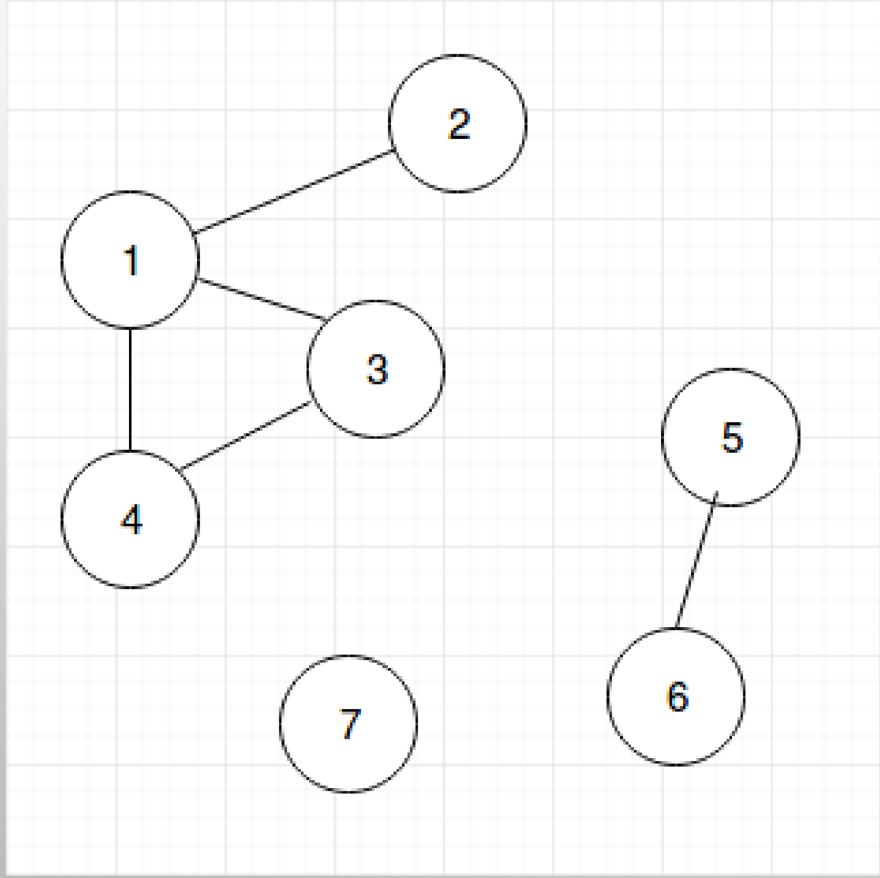


- For $K_{3,3}$ we have
  $e = 9$ & $n = 6$

But $e = 9 \geq 2n - 4 = 2.6 - 4 = 8$
Lemma – 2 fails.

- <u>What can you say about any complete bipartite graph $K_{m,n}$ with $m, n \geq 3$?</u>

Extension of Euler's Formula – If G is any planar graph, then n(G) - e(G) + f(G) = k + 1, where k is the number of components in the graph.



- n = 7, e = 5, f = 2 & k = 3

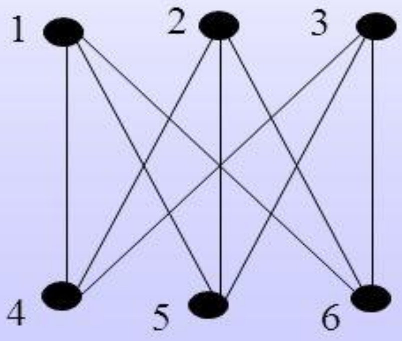
  n – e + f = 7 – 5 +2 = 4

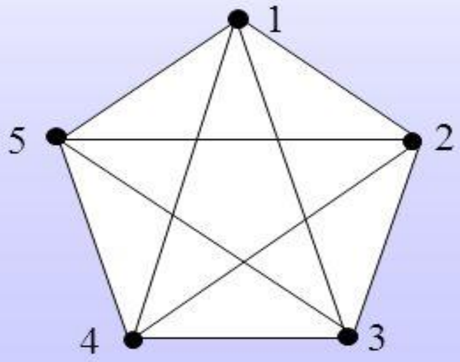
  k + 1 = 3 + 1 = 4

Hence theorem verified.

Theorem 1 (Kuratowski, 1930)

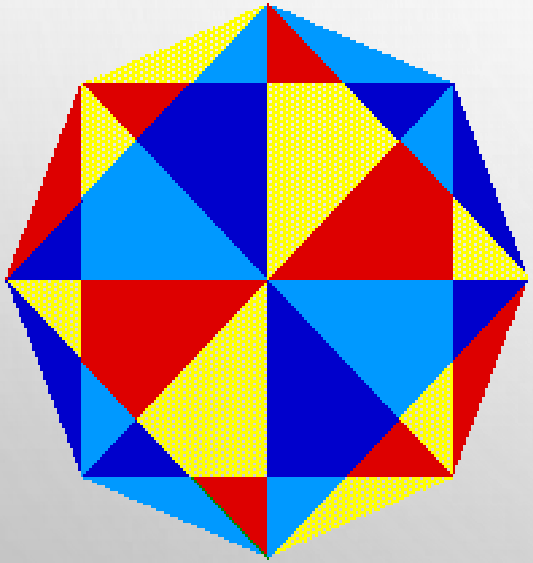
- A graph is planar if and only if it does not contain a subgraph that is $K_{3,3}$ or K_5 configuration.
  - Recall: K_n is a graph on *n* vertices with an edge joining every pair of vertices.(K_3 is a triangle.) $K_{m,n}$ is a bipartite graph with *m* and *n* vertices in its two vertex sets and all possible edges between vertices in the two sets.

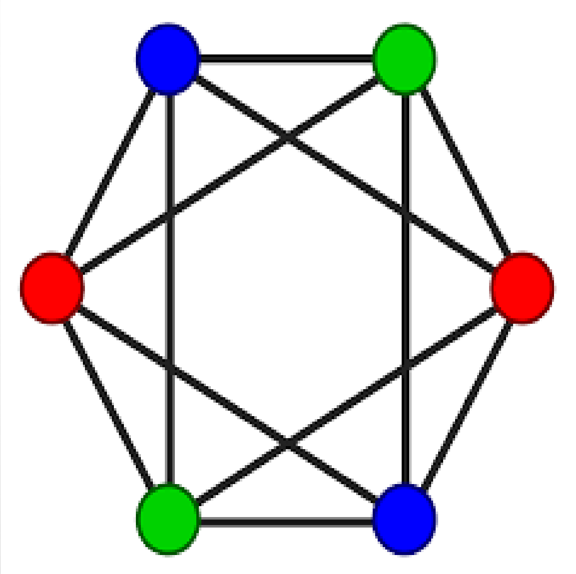


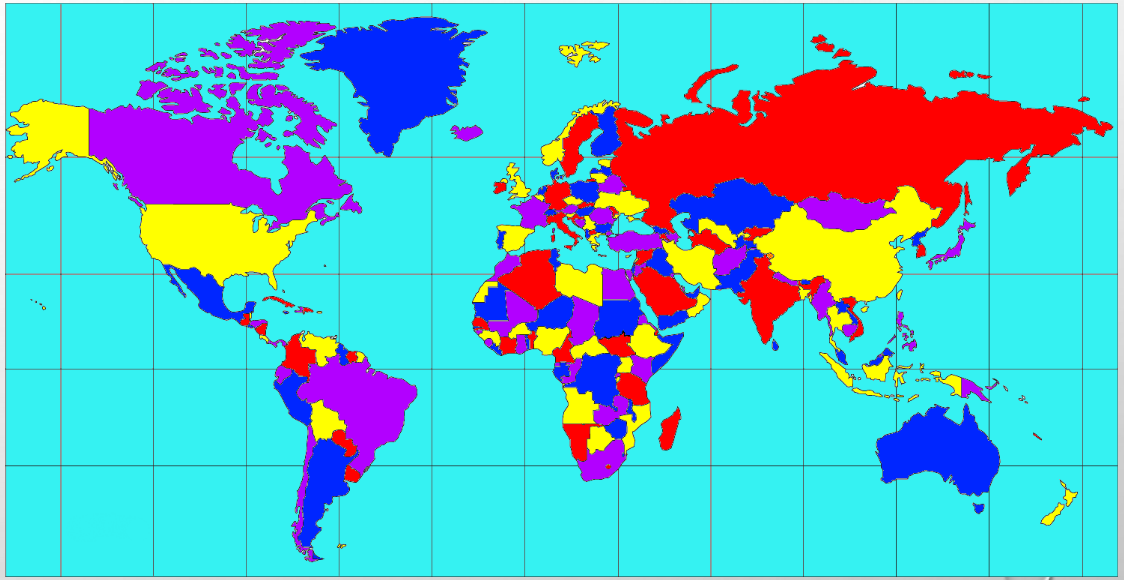


Graph Coloring

- ***Given any map of countries, states, etc., how many colors are needed to color each region on the map so that neighboring regions are colored differently?***







- A Coloring of a simple graph is the assignment of a color to each vertex of the graph such that **no two adjacent vertices or no two adjacent edges** are assigned the same color.

❑ Method to Color a Graph –

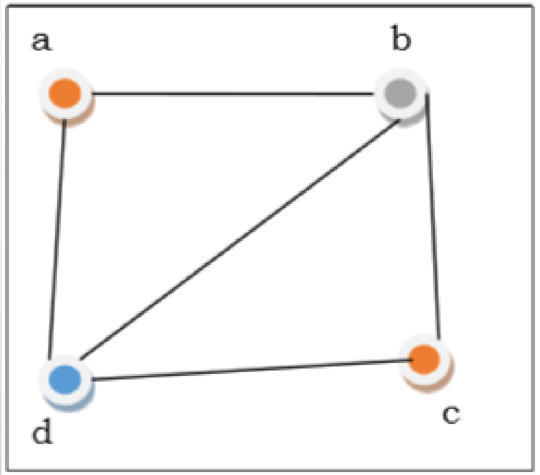
The steps required to color a graph G with n number of vertices are as follows −

Step 1 − Arrange the vertices of the graph in some order.

Step 2 − Choose the first vertex and color it with the first color.

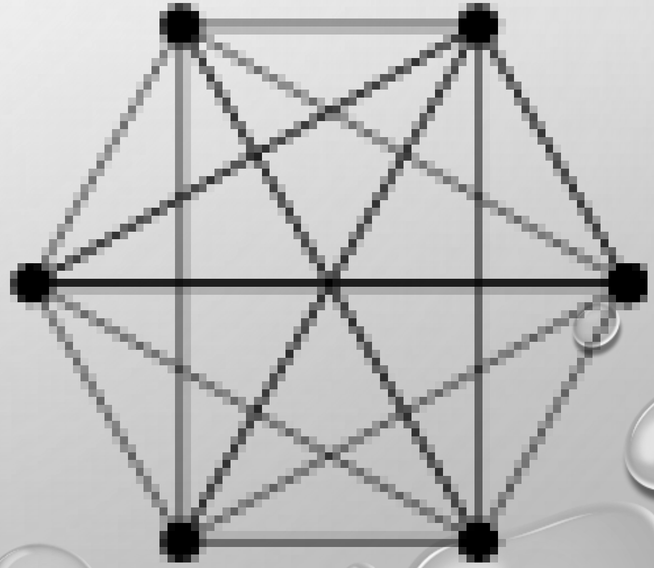
Step 3 − Choose the next vertex and color it with the lowest numbered color that has not been colored on any vertices adjacent to it. If all the adjacent vertices are colored with this color, assign a new color to it. Repeat this step until all the vertices are colored.

Chromatic number – The least number of colors required to color a graph is called its chromatic number. It is denoted by $\chi(G)$.



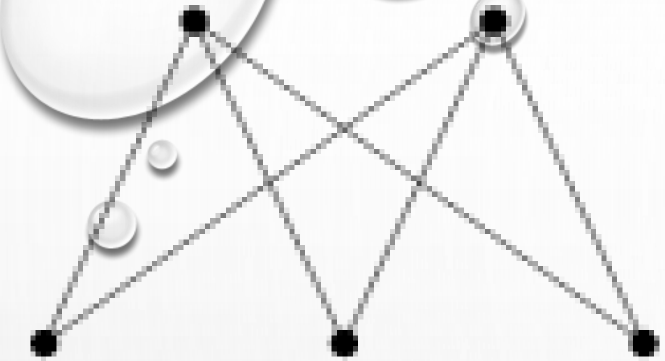
- In the figure, at first vertex a is colored red. As the adjacent vertices of vertex a are again adjacent to each other, vertex b and vertex d are colored with different color, black and blue respectively. Then vertex c is colored as red as no adjacent vertex of c is colored red. Hence, we could color the graph by 3 colors.

- Hence, the chromatic number of the graph is 3.

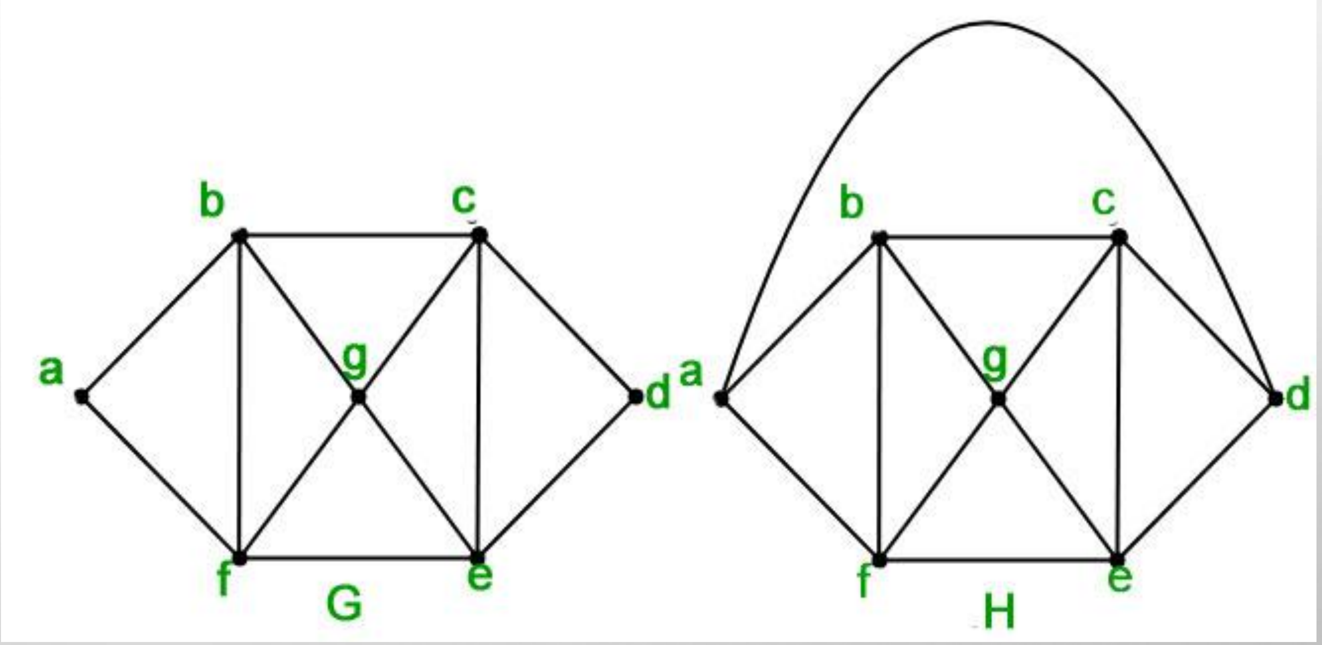


- The graph on the right is K_6, a complete graph with 6 vertices. The only way to properly color the graph is to give every vertex a different color (since every vertex is adjacent to every other vertex).

- Thus the chromatic number is 6.



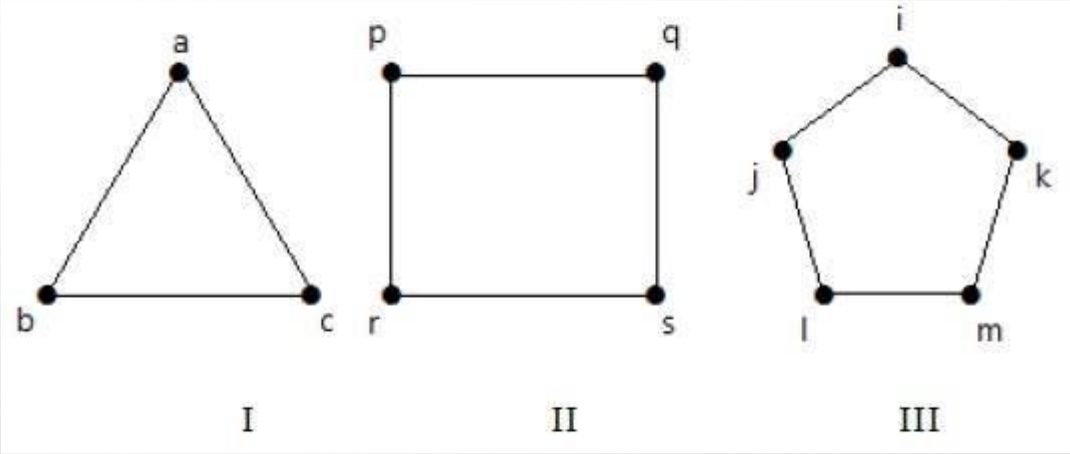
- This graph is just another bipartite graph $K_{2,3}$. Color the vertices on the top row red and the vertices on the bottom row blue. As with all bipartite graphs, this graph has chromatic number 2.



- In graph G, the chromatic number is at least 3 since the vertices a, b, and f are connected to each other.
- In graph H since a and d are also connected, therefore the chromatic number is 4.

Chromatic numbers of C_n , K_n , $K_{m,n}$

Graph	**Chromatic Number**
K_n, complete graph	n
$K_{m,n}$, complete bipartite graph	2
C_n, cycle of length n	2, if n is even
	3, if n is odd



- ***Four Color Theorem - If G is a planar graph, then the chromatic number of G is less than or equal to 4. Thus any map can be properly colored with 4 or fewer colors.***

- Five color theorem - Given a plane separated into regions, such as a political map of the counties of a state, the regions may be colored using no more than five colors in such a way that no two adjacent regions receive the same color.

- Upper and Lower bounds for chromatic numbers - for every graph G, the chromatic number of G is at least 1 and at most the number of vertices of G.

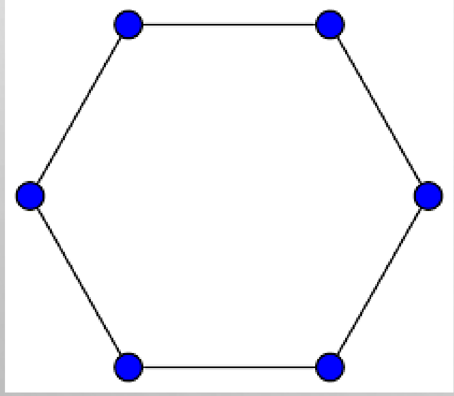
- Let $\Delta(G)$ be the largest degree of any vertex in the graph G. One reasonable guess for an upper bound on the chromatic number is $\chi(G) \leq \Delta(G)+1$.

- **Brook's Theorem** - *Any graph* G *satisfies* $\chi(G) \leq \Delta(G)$, *unless* G *is a complete graph or an odd cycle, in which case* $\chi(G) = \Delta(G)+1$.

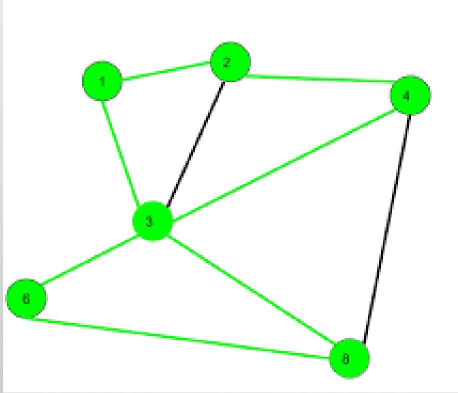
Clique Number, $\boldsymbol{\omega(G)}$

- **Clique** - A **clique** in a graph is a set of vertices all of which are pairwise adjacent. In other words, a clique of size n is just a copy of the complete graph K_n.

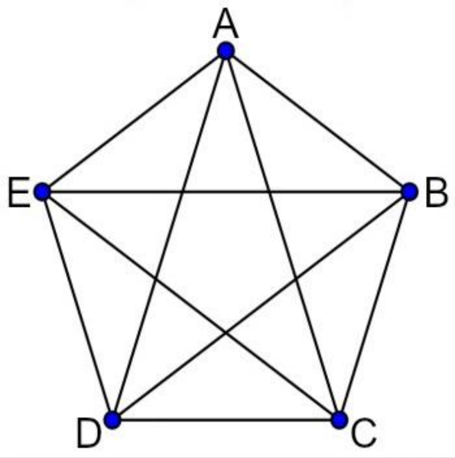
- We define the **clique number, $\boldsymbol{\omega(G)}$**, of a graph to be the largest n for which the graph contains a clique of size n.



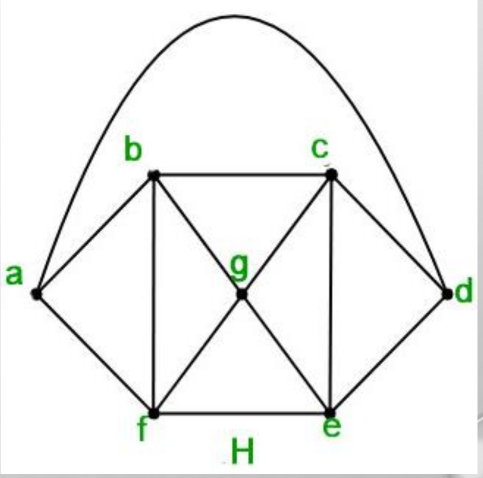
$\boldsymbol{\omega(G)} = 2$



$\boldsymbol{\omega(G)} = 3$



$\boldsymbol{\omega(G)} = 5$

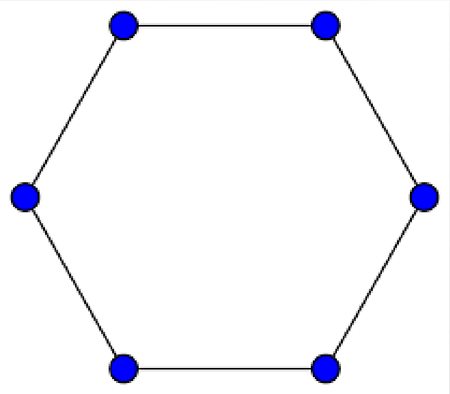


$\boldsymbol{\omega(H)} = 3$

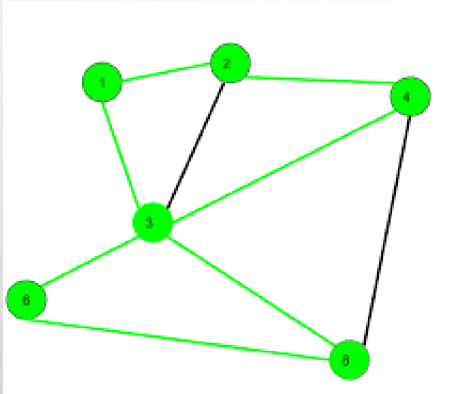
➢ Relation between χ(G) & $\omega(G)$ -

- ❑ Theorem - ***<u>The chromatic number of a graph G is at least the clique number of G i.e. χ(G) ≥ $\omega(G)$.</u>***

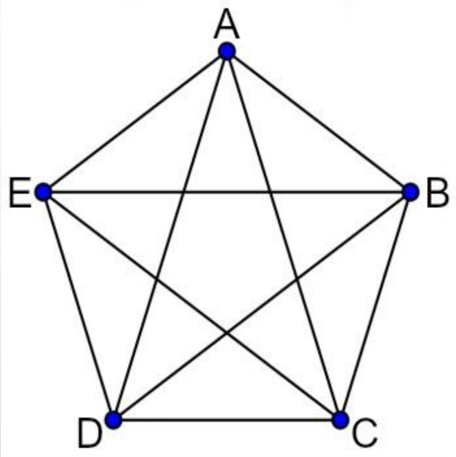
- ❑ Any clique of size n cannot be colored with fewer than n colors, so we have a nice lower bound.



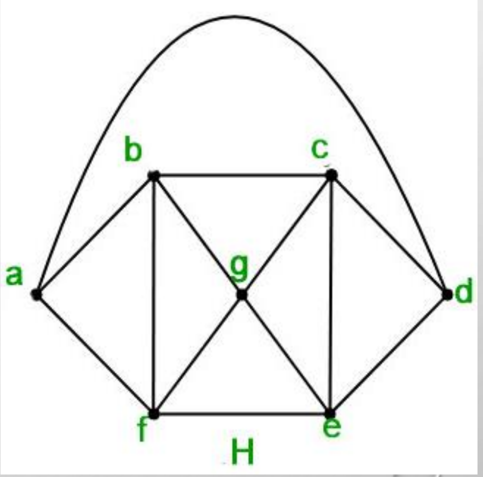
$\omega(G) = 2$
χ(G) = 2



$\omega(G) = 3$
χ(G) = 3



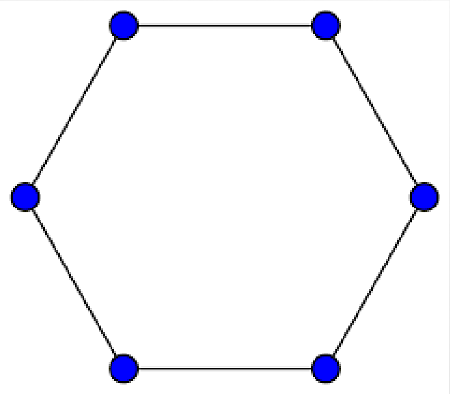
$\omega(G) = 5$
χ(G) = 5



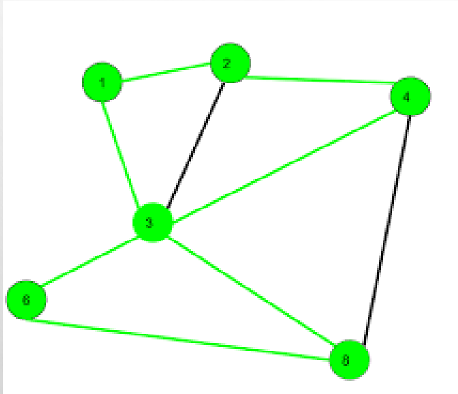
$\omega(H) = 3$
χ(G) = 4

Perfect Graph

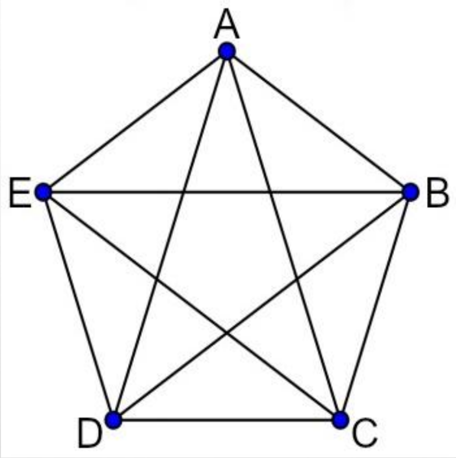
- *There are times **when the chromatic number of G is equal to the clique number. These graphs have a special name; they are called perfect.** If you know that a graph is perfect, then finding the chromatic number is simply a matter of searching for the largest clique. However, not all graphs are perfect.*



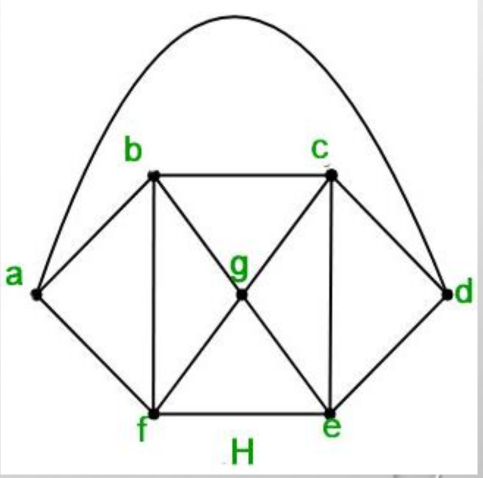
$\omega(G)$ = 2
χ(G) = 2
Perfect Graph



$\omega(G)$ = 3
χ(G) = 3
Perfect Graph

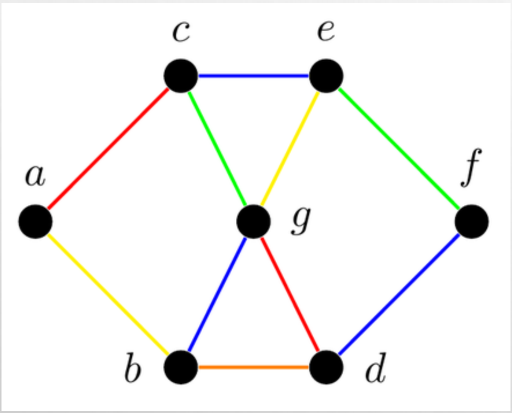


$\omega(G)$ = 5
χ(G) = 5
Perfect Graph

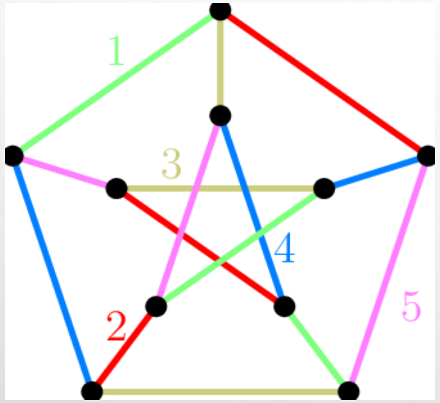


$\omega(H)$ = 3
χ(G) = 4
Non-Perfect Graph

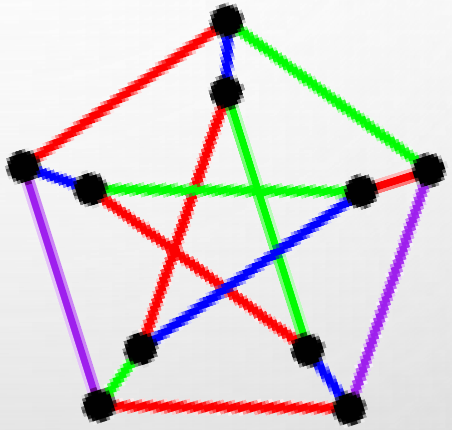
- **Coloring Edges - E**dges that are adjacent must be colored differently. Here, we are thinking of two edges as being adjacent if they are incident to the same vertex.



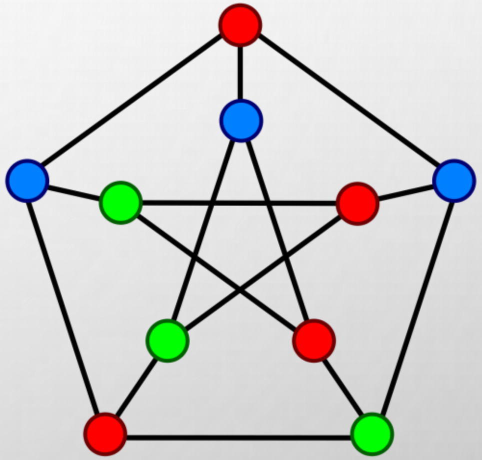
$\chi'(G) = 3$



Petersen Graph



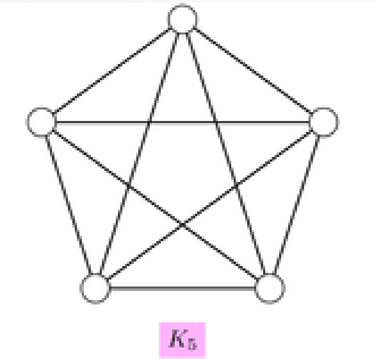
$\chi'(G) = 4$

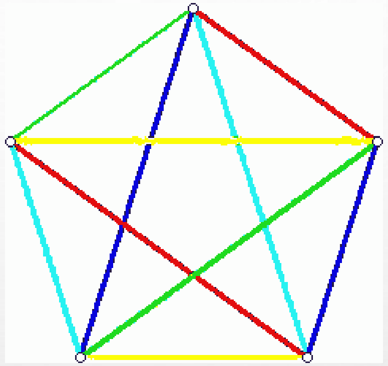


$\chi(G) = 3$

- The least number of colors required to properly color the edges of a graph G is called the **chromatic index** of G, written $\chi'(G)$.

- Chromatic number of K_5 is 5 and the chromatic index is also 5.



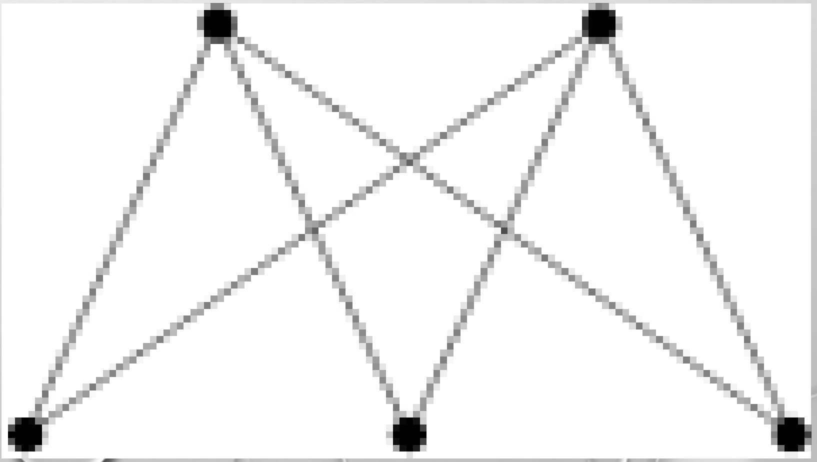


- So in general, what can we say about chromatic index? Certainly $\chi'(G) \geq \Delta(G)$. But how much higher could it be?

- **Vizing's Theorem -** *For any graph* G, *the chromatic index* $\chi'(G)$ *is either* $\Delta(G)$ *or* $\Delta(G)+1$.

- **Brook's Theorem -** *Any graph* G *satisfies* $\chi(G) \leq \Delta(G)$, *unless* G *is a complete graph or an odd cycle, in which case* $\chi(G) = \Delta(G)+1$.

Chromatic Number : $\chi(G) = 2$
Chromatic Index : $\chi'(G) = 3$



Thank You