8.6 Applications of Inclusion-Exclusion

Many counting problems can be solved using the principle of inclusion-exclusion. The famous hat-check problem can be solved using the principle of inclusion-exclusion. This problem asks for the probability that no person is given the correct hat back by a hat-check person who gives the hats back randomly.

An Alternative Form of Inclusion-Exclusion

There is an alternative form of the principle of inclusion-exclusion that is useful in counting problems. In particular, this form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \ldots, P_n$.

Let A_i be the subset containing the elements that have property P_i. Let's denote the number of elements with all the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ by $N(P_{i_1}P_{i_2}\ldots P_{i_k})$. Writing these quantities in terms of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = N(P_{i_1}P_{i_2}\ldots P_{i_k}).$$

Let's denote the number of elements with none of the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ by $N(P'_{i_1}P'_{i_2}\ldots P'_{i_k})$. Suppose the number of elements in the set is N. Then it follows that

$$N(P'_{i_1}P'_{i_2}\ldots P'_{i_k}) = N - |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}|.$$

From the inclusion-exclusion principle, we see that

$$N(P'_{i_1}P'_{i_2}\ldots P'_{i_k}) = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) - \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2\ldots P_n).$$

Example 1. *How many solutions does*

$$x_1 + x_2 + x_3 = 11$$

have, where $x_1, x_2,$ and x_3 are nonnegative integers with $x_1 \le 3, x_2 \le 4,$ and $x_3 \le 6$?

Solution. To apply the principle of inclusion-exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \le 3, x_2 \le 4$, and $x_3 \le 6$ is

$$N(P'_1P'_2P'_3) = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3).$$

Using the same techniques as in Section 6.5, it follows that

- $N =$ the total number of solutions with $x_1, x_2, x_3 \in \mathbb{N} = \binom{11+3-1}{11} = \binom{13}{11} = 78$,
- $N(P_1) =$ number of solutions with $x_1 > 3$ (so $x_1 \ge 4$) $= \binom{7+3-1}{7} = \binom{9}{7} = 36$,
- $N(P_2) =$ number of solutions with $x_2 > 4$ (so $x_2 \ge 5$) $= \binom{6+3-1}{6} = \binom{8}{6} = 28$,
- $N(P_3) =$ number of solutions with $x_3 > 6$ (so $x_3 \ge 7$) $= \binom{4+3-1}{4} = \binom{6}{4} = 15$,
- $N(P_1P_2) =$ number of solutions with $x_1 \ge 4$ and $x_2 \ge 5 = \binom{2+3-1}{2} = \binom{4}{2} = 6$,
- $N(P_1P_3) =$ number of solutions with $x_1 \ge 4$ and $x_3 \ge 7 = \binom{0+3-1}{0} = \binom{2}{0} = 1$,
- $N(P_2P_3) =$ number of solutions with $x_2 \ge 5$ and $x_3 \ge 7 = 0$,
- $N(P_1P_2P_3) =$ number of solutions with $x_1 \ge 4, x_2 \ge 5$ and $x_3 \ge 7 = 0$.

Inserting these quantities into the formula for $N(P'_1P'_2P'_3)$ shows that the number of solutions with $x_1 \le 3, x_2 \le 4$, and $x_3 \le 6$ equals

$$N(P'_1P'_2P'_3) = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6.$$

□

Derangements

We may use the principle of inclusion-exclusion to count the permutations of n objects that leave no objects in their original positions.

Example 2 (The Hat-check Problem)**.** *A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?*

Remark 1. *The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats. We will return to this example after we find the number of permutations of n objects that leave no objects in their original position.*

Definition 1. *A derangement is a permutation of objects that leaves no object in its original position.*

To solve the hat-check problem, we will need to determine the number of derangements of a set of n objects.

Example 3. *The permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345, because this permutation leaves 4 fixed.*

Notation. *Let D_n denote the number of derangements of n objects.*

Example 4. *If $n = 3$, then $D_3 = 2$, because the derangements of 123 are 231 and 312.*

We will evaluate D_n, for all positive integers n, using the principle of inclusion-exclusion.

Theorem 1. *The number of derangements of a set with n elements is*

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!}\right].$$

Proof. Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$. This means that

$$D_n = N(P_1'P_2'\ldots P_n').$$

Using the principle of inclusion-exclusion, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_iP_j) - \sum_{i<j<k} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2\ldots P_n),$$

where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fix at least three elements, and so on. We will now find all the quantities that occur on the right-hand side of this equation.

First, note that $N = n!$, because N is simply the total number of permutations of n elements. Also, $N(P_i) = (n-1)!$. This follows from the product rule, because $N(P_i)$ is the number of permutations that fix element i, so the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_iP_j) = (n-2)!,$$

because this is the number of permutations that fix elements i and j, but where the other $n-2$ elements can be arranged arbitrarily. In general, note that

$$N(P_{i_1}P_{i_2}\ldots P_{i_m}) = (n-m)!,$$

because this is the number of permutations that fix elements $i_1, i_2, \ldots, i_m$, but where the other $n-m$ elements can be arranged arbitrarily. Because there are $\binom{n}{m}$ ways to choose m elements from n, it follows that

$$\sum_{1 \le i \le n} N(P_i) = \binom{n}{1}(n-1)!,$$

$$\sum_{1 \le i < j \le n} N(P_i P_j) = \binom{n}{2}(n-2)!,$$

and in general,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_m \le n} N(P_{i_1} P_{i_2} \ldots P_{i_m}) = \binom{n}{m}(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n}(n-n)! \\ &= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!0!}(n-n)!. \end{aligned}$$

Simplifying this expression gives

$$D_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \cdot \frac{1}{n!} \right]. \qquad \square$$

We may now use the formula from the above theorem to find D_n for a given positive integer n. For instance,

$$D_3 = 3! - 3! + \frac{3!}{2!} - \frac{3!}{3!} = 6 - 6 + 3 - 1 = 2.$$

Solution to the hat-check problem. The probability that no one receives the correct hat is $\frac{D_n}{n!}$. By Theorem 2, this probability is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}.$$

Using methods from calculus it can be shown that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} + \cdots \approx 0.368.$$

Because this is an alternating series with terms tending to zero, it follows that as n grows without bound, the probability that no one receives the correct hat converges to $e^{-1} \approx 0.368$. In fact, this probability can be shown to be within $\frac{1}{(n+1)!}$ of e^{-1}. $\square$